Reputation, Quality Choices and Optimal Product Discontinuation Policy

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Abstract

This paper studies quality choice and optimal product discontinuation decisions when a firm's intended quality choice is difficult to control and information about actual quality is noisy. The market forms beliefs about the actual quality based on the observed sequence of noisy signals or product's reputation. It is shown that optimal product discontinuation policy exists. A firm discontinues its product when reputation or realised quality falls below a threshold. Incentives for providing higher quality arise from two channels: higher quality increases (i) consumers' valuation for the product conditional on its being continued and (ii) the product's chance of surviving the policy. The firm exerts higher quality and applies a more stringent reputation standard discontinuation policy when incentives from both channels are strengthened. This happens when either the firm cares more about the long-term relationship or the precision of the initial prior belief decreases. Signal informativeness has ambiguous effects on the optimal discontinuation policy as its effects on consumers' valuation are unclear. However, without discontinuation policy, higher informativeness unambiguously increases quality.

Key words: Consumer Learning, Optimal Stopping, Product Discontinuation.

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Character is like a tree and reputation like its shadow. The shadow is what we think of it; the tree is the real thing.

–Abraham Lincoln

1 Introduction

It has long been known that imperfect information can cause a wide range of market imperfections. One example that has been widely discussed is market failure when the product's quality is unobservable. This is particularly the problem where consumers are only able to evaluate the quality of the products after they have been bought and consumed. These products are known as "experience goods". Examples include durables such as cars or used cars, electronic appliances, nondurables such as wine, and services such as airlines or specialist professions. Usually, the problem occurs even in a market where the product's characteristics are observable or information on inherited quality is available to the market but a firm cannot perfectly control the intended quality then the true product's quality is unknown to the market and consumers' utility derived from the products requires a match between product characteristics or features and consumer's taste or preference. For instance, technology gadgets such as the mobile phone: information on its features (camera resolution, 3G compatibility, lightness of weight) is available, but all the features available on it may be rarely used by consumers and they may find these out only after purchase. Fashion or designer branded products are another example: the quality of the materials may be observable, but the consumer may not like them because they may not match the consumer's preference or taste. Many mechanisms have been proposed to solve this problem. One possible way is by repeated interactions, where the consumer frequently purchase the product and is able to observed its realised quality, although still not able to observe the firm's efforts in quality. In these settings, the consumer's experiences with a particular product provide imperfect information on a combination of the firm's efforts, a suiting of their tastes, and sheer luck. This collection of experiences forms an expectation of quality, and is the definition of reputation for this paper.

As the learning process continues, information on the quality of the product is gradually revealed to the market. Many producers, after observing the sequence of the product's quality reputation over some periods, decide to take the product out of the market and launch a new one before the quality of the old product is completely revealed. The market then has to start the learning process over again. These "discontinuation policies" typically are deterministic or fixed in advance, for example, cars where Japanese car manufacturers have a policy of launching a new car design approximately every 5 years while German manufacturers do this every 7 years, and designer clothes where a new fashion is introduced every season. However, some firms adopt a discontinuation policy that is not based on the length of time that the product has been on the market. Different firms may apply different criteria for discontinuing their products. Some products such as computer processors are discontinued because of the advancement of new technology. Some firms launch a new product in response to their competitors. For example, Sony was planning to launch its new generation game console Play Station 3 (PS3) in the year 2007. However, after its top-of-the-line rival Microsoft's Xbox 360 was released in November 2005, Sony revised their PS3 launching plan to the spring of 2006. Nevertheless, because of a technical problem, the PS3 was finally released in November 2006. Some firms use their product's quality deteriorating reputation as a criterion for discontinuing. A good example is the series of battery recalls from many computer manufacturers such as Lenovo; this manufaturer announced a recall of 526,000 laptop batteries on September 2006, after a Lenovo ThinkPad T43 caught fire at the Los Angeles International Airport. Or the notebook battery recalls issued by Dell and Apple Computer in August 2006, after it was discovered that some batteries were overheating or exploding, causing fires even when the machines were turned off.[1] It is interesting to see how firms optimally determine their discontinuation policy when information about product quality is imperfect and how the policy changes when the informativeness of consumers' experiences changes.

This paper studies quality choice and product discontinuation problems under a moral hazard when information about product quality is imperfect but symmetric; information about the true product quality is unknown to both consumers and firm. The firm's intended quality choice is difficult to control and is subject to a random shock. The paper focuses in particular on product discontinuation policies when the firm uses quality reputation as a criterion of discontinuation.

The paper is organized as follows. Section 2 reviews the related literature. Section 3 sets up the basic model and discusses some results and implications. Section 4 introduces the problem of product discontinuation by considering the case where all products have a predetermined time in the market before being discontinued. Section 5 analyses the

[1] http://news.zdnet.co.uk/hardware/0,1000000091,39283811,00.htm

discontinuation problem when the product stopping period is stochastic and shows that an optimal discontinuation policy exists. The optimal policy is then derived and comparative static analyses are carried out. Section 6 presents the conditions for optimal quality choice under optimal product discontinuation policy, and discusses how a product discontinuation policy can affect a firm's incentive to provide quality. Section 7 provides conclusions.

2 Literature Review

Reputation has been extensively studied in economics as a mechanism that could alleviate inefficiencies arising from imperfect information. The notions of reputation in economics can be categorized into two mechanisms (Cabral 2005). The first is where the firm's or agent's action is unobservable; a moral hazard problem. Reputation may arise either from repeated interactions between a firm and consumers or be an implicit contract between two parties. The second involves adverse selection problems where there are information asymmetries between the firm and the consumers. Quality is a characteristic or type of firms that is known exclusively to them. In this case, uninformed consumers have access to the history of past stage outcomes and form their poseterior beliefs on this regarding the firm's type. The informed firm can improve its long-run profit by acting in a way that will convince consumers to believe that it belongs to the better type.

For reputation mechanisms under moral hazard, the seminal work of Klien and Leffler (1981) suggests that reputation mechanisms can ensure high quality performance in circumstances where quality is not observable. In order to sustain a high reputation and produce high quality goods, the returns to the firm must be high and persuasive enough to offset a short-run incentive to deviate. The idea is explored more formally in Shapiro (1983), where different firms specialise in different quality products and entry and exit are incorporated. Reputation is defined as the consumer's expectation of quality, and it has been shown that reputation can work as an imperfect mechanism for quality assurance and that high quality products are sold at a premium above their cost. Both papers' analyses are based on the assumption that the experience from consuming the product delivers the perfect signal about quality. The consumers knows the quality completely and immediately after consumption of the product.[2] Rogerson (1983) considers the role of reputation in assuring quality where consumers are able to observe quality after purchase imperfectly

[2] Dybvig and Spatt (1982) and von Weizsäcker (1980) apply a similar idea of a perfect signal of quality, but consumers become aware of this with a lag.

and probabilistically.

An idea similar to that of Klien and Leffler (1981) appears in labour economic literature; Fama (1980) explains situations where the labour market does not know the agent's true ability or talent but learns this from observing past performances. However, agents are disciplined by implicit contracts or their career concerns; the market pays according to its belief about an agent's ability. In other words, an agent's compensation depends upon the market's belief of his ability and reputation refers to this belief. This idea is formalised by Holmström (1982). In this celebrated paper, information about an agent's ability is imperfect but symmetric, i.e. it is unknown to both agent and employers. Quality produced by the agent is a combination of the agent's talent, effort and luck and is a noisy signal to actual ability. Holmström (1982) shows that, at the beginning of a relationship where an agent's talent is unknown, the agent exerts high effort to produce high quality. As time progresses, the market updates the belief about the agent's ability and, finally, this will be completely realised. Efforts, therefore, decrease over time. Holmström (1982)'s reasoning shows that no equilibrium in which a single firm commits to a high quality product exists.

As noted by Mailath and Samuelson (2006), reputations are temporary if the type of the player is determined once and for all at the beginning of the game. Reputations can have long-run implication by incorporating some mechanisms by which the uncertainty about types is continually replenished. Examples include Holmström (1982), Cole et al (1995), Mailath and Samuelson (2001), and Phelan (2001) where the type is governed by a stochastic process.

An alternative reputation mechanism under adverse selection is pioneered by Kreps and Wilson (1982), Milgrom and Roberts (1982) and, Kreps et al. (1982). The notion of a "commitment" type[3], long-run players who always play the same action, is introduced. An important commitment type is the Stackelberg type. The Stackelberg type is a long-run players committing to the Stackelberg action. As long as the future returns from maintaining a reputation from playing the Stackelberg action exceeds short-run returns from playing other actions, the equilibrium strategy for an ordinary player will be to try to acquire a reputation by masquerading as a Stackelberg type . This issue is treated in detail in Mailath and Samuelson (2006).

Hörner (2002) presents a model where competition can generate a reputation mechanism under adverse selection through repeated interactions between firms and consumers.

[3] Also called "Irrational", "Action", "Behavioural" or "Crazy" types.

As in Holmström (1982), quality is a noisy signal of a firm's efforts in this model. However, Hörner (2002) shows that competition generates an outside option for consumers to leave a firm when they believe that the firm belongs to a bad type. This poses a threat to a firm and induces it to constantly exert high efforts.

There has been a growing body of literature focusing on the economics of brands and reputations. Brands are viewed as carriers of reputations and can be traded. Tadelis (1999) explores this idea and formulates a basic framework to analyse the market of reputation. His paper shows that this market is indeed sustainable. In addition, it provides all agents both "old" and "young" with equal incentives to exert effort (Tadelis 2002). However, Mailath and Samuelson (2001) show that the long-run effects of such a market are complicated as good reputations are likely to be purchased and depleted by "inept" agents. Kreps (1990) presents an argument for trading reputations in a moral hazard type of model.

Cabral (2000) adopts the framework from Tadelis (1999) to study the brand stretching decision of a firm when it launches a new product. The paper shows that brand stretching signals high quality; better firms tend to launch new products under the same brand.

The theory of optimal stopping has been applied in many areas of economics. For example, Dutta and Rustichini (1993) study optimal stopping games, a variant of stochastic games, where each player has an irreversible action. The paper shows the existence of stopping equilibria and presents two applications of product innovations and asset sales. Feng and Gallego (1995) study the optimal time for a single price change from a given initial price. The paper derives conditions under which the firm should decrease or increase the price. This can be considered as time optimal time to start an "end-of-season" sale or stop promotional pricing.

However, the problem studied in this paper is slightly different from the standard single time stopping problem. The model presented in the paper allows the firm to stop the product and start a new one. Therefore, any optimal stopping (or starting) rule[4] must account for this. In operation research literature, this type of problem is called a restart problem according to Katehakis and Vienott (1987). It has been shown by Gittins and Jones (1974), Whittle (1980,1982) and Variaya et al. (1985) that this sequence of a one-dimensional stopping problem (where only one project or product is being considered in each period) is much simplified but equivalent to the multi-armed bandit problem (the problem where at each stage there are many choices of possible actions or experiments,

[4]In this thesis, the words discontinuation and stopping or policy and rule are used interchangeably.

and a choice of action j results in an observation being taken from a jth experiment; the goal is to maximise the present value of an infinite stream of rewards).

3 The Basic Quality Choice Model with Consumer Learning

The basic model presented in this section is adopted and simplified from Holmström (1982)'s career concerns model. The difference is that, in Holmström (1982), an agent (a firm) chooses an action (quality choice) in every period but in this model product's quality is decided by a firm only once at the initial period and remains unchanged until the product is discontinued. Another major extension, which will be considered in later sections is the incorporation of options for a firm to discontinue a product and relaunch a new one. Therefore, the model in this section provides the basis for this extension.

Consider a simple discrete time model where a firm offers a single product for sale in a market consisting of a large number of identical consumers. At the beginning of the first period, a firm chooses once and for all its desired level of product quality, q. However, the actual product quality is $\tilde{q} = q + \gamma$ where γ is an idiosyncratic quality disturbance term and can be viewed as a random term that determines how the product matches the consumer's taste. The firm wants to produce a good quality product and satisfy consumer needs, but some characteristics of the product may not fit well with consumer's taste or may not be what the consumer actually wants. Therefore, the actual product quality is a combination of the true or intrinsic quality that the firm produces and a match with the consumer's taste. Example of this type of product includes brand name clothes where fashion designers produce good quality clothes with styles that may or may not match the consumer's taste. Other examples includes product with sophisticated technology and features such as cars, mobile phones or laptop computers. These products come with many features and vary in quality. Consumers will learn whether every feature suits with their preference when the products is purchased and used. It is assumed that γ is normally distributed with a zero mean and variance σ_q^2.

Consumers are identical and live for only one period. In each period, new consumers come to the market and decide whether to buy a firm's product and how much they are willing to pay for it. The quality perceived by consumers is a noisy signal of the actual quality. If consumers decide to buy the product from the firm, the utility gained by them is $u_t = \tilde{q} + \varepsilon_{u_t}$ where $\varepsilon_{u_t} \sim N\left(0, \sigma_u^2\right)$. The term ε_{u_t} can be interpreted as the consumers' random experience of using the product. It reflects the fact that consumers with identical taste using identical products can gain different utility levels depending on some random factors such as how the product is used and taken care of, knowledge of the product's specifications, and familiarity with the product's features and functions. Having realised the utility gained, consumers make an imperfect report of their experience, $r_t = u_t + \varepsilon_{r_t}$ where $\varepsilon_{r_t} \sim N\left(0, \sigma_r^2\right)$. The report is imperfect because consumers cannot precisely communicate their feeling about the product to the next generation of consumers. The sequence of reports, $\{r_s\}_1^t$, and, therefore, the firm's average quality report or *reputation,* $R_t = \frac{\sum_{s=1}^t r_s}{t}$, are publicly observable and, since consumers cannot directly observe the actual quality of the product, $\tilde{q}$, they use this information to decide how much they are willing to pay for the product.

In any period t, after observing R_{t-1}, consumers are willing to pay up to their expected utility from the product. That is,

$$\begin{aligned} p_t &= E\left[u_t \,|R_{t-1}\right] \\ &= E\left[q \,|R_{t-1}\right] + E\left[\gamma \,|R_{t-1}\right]. \end{aligned} \tag{3.1}$$

The firm is risk neutral choosing quality level q^* to maximise the expected sum of profit over product life T.

$$q^* = \arg\max_q \sum_{t=1}^{T} \delta^{t-1}\left[E\left[p_t\right] - C\left(q\right)\right] \tag{3.2}$$

where $\delta \in (0,1)$ is a discount factor[5], $C(q)$ is the unit cost of producing a product with quality q, and $C(q)$ is increasing and convex in q. Note that, even though the market is not able to observe the firm's choice of quality, q, directly, it can be inferred by solving (3.2). Therefore, at period $t+1$, observing r_t in equilibrium will be equivalent to observing the sequence

$$\begin{aligned} z_t \equiv r_t - q^* &= q + \gamma + \varepsilon_{u_t} + \varepsilon_{r_t} - q^* \\ &= q + \gamma + \varepsilon_t - q^* \end{aligned} \tag{3.3}$$

At equilibrium $q = q^*$, then

$$z_t = \gamma + \varepsilon_t \tag{3.4}$$

[5]It is assumed that δ cannot be equal to 1; to avoid complication that may arise in defining value function for an optimal product stopping problem, see Section 5.1.

where the composite noise, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2 = \sigma_u^2 + \sigma_r^2)$, is the combination of two error terms, $\varepsilon_{u_t} + \varepsilon_{r_t}$.

Consumers and the firm share a prior belief about γ. The prior distribution is assumed normally distributed with mean m_1 and variance σ_1^2. Define the precision of ε_t as $h_\varepsilon = 1 \big/ \sigma_\varepsilon^2$ and precision of belief at time t as $h_t = 1 \big/ \sigma_t^2$ where σ_t^2 denotes variance of prior belief about γ at time t. The posterior distribution of γ will still be normal distribution with mean, variance and precision given by

$$m_{t+1} = \frac{h_t m_t + h_\varepsilon z_t}{h_t + h_\varepsilon} = \frac{h_1 m_1 + h_\varepsilon \sum_{s=1}^{t} z_s}{h_1 + t h_\varepsilon} \tag{3.5}$$

$$h_{t+1} = h_\varepsilon + h_t = t h_\varepsilon + h_1 \tag{3.6}$$

Consumers' willingness to pay (3.1) can be rewritten as

$$p_t = E\left[u_t \,|R_{t-1}\right] = q^* + E\left[\gamma \,|R_{t-1}\right] = q^* + m_t \tag{3.7}$$

The firm's problem is then to attain the supremum of

$$\sup_q \sum_{t=1}^{T} \delta^{t-1} \left[E\left[p_t\right] - C\left(q\right)\right]$$

or equivalently,

$$\sup_q \sum_{t=1}^{T} \delta^{t-1} \left[q^* + E\left[m_t\right] - C\left(q\right)\right] \tag{3.8}$$

using (3.5) and (3.3), the above expression becomes

$$\sup_q \sum_{t=1}^{T} \delta^{t-1} \left[q^* + \frac{h_1 m_1}{h_t} + \frac{h_\varepsilon}{h_t} \sum_{s=1}^{t-1} \left(E(\gamma) + q - q^*\right) - C\left(q\right)\right]$$

using (3.6) to get,

$$\sup_q \sum_{t=1}^{T} \delta^{t-1} \left[q^* + \frac{h_1 m_1}{h_t} + \frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} \left(m_1 + q - q^*\right) - C\left(q\right)\right] \tag{3.9}$$

The solution to (3.9) is then given by the first order condition (FOC)

$$\sum_{t=1}^{T} \delta^{t-1} \left[\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} - C'(q^*) \right] = 0 \tag{3.10}$$

or,

$$\frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} \left[\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} \right] = C'(q^*) \tag{3.11}$$

From (3.11), the optimal level of quality depends upon the discount factor, the structure of the cost function, the precision of the random noise (the sum of the perceived quality noise and the imperfect report) and the precision of the first period prior belief as well as the product lifespan. When the firm becomes more patient or δ increases, it places greater weight on future benefits and futures become more important; the incentive to bias consumers' learning increases, and the more quality is exerted.

As the precision of the first period prior belief, h_1, increases, the left-hand side (LHS) of (3.11) decreases. Since the convex cost structure is assumed, the firm will choose a lower level of quality in equilibrium. The reason for this is that, since the true product's quality is unobservable, consumers have to form beliefs about the product. A higher level of perceived quality influences perceptions about the quality disturbance term. The firm has an incentive to jam the signal by choosing higher quality in which it would potentially bias the process of the inference of consumers. As the precision of prior belief increases, this implicit incentive decreases and the lower product quality is optimal.

Recall that the composite error term, ε_t, is the sum of two idiosyncratic terms: consumers' utility noise, ε_{u_t}, and noise on consumer report, ε_{r_t}. The higher the variances of these two noise terms, the higher the variance of the composite error term, and hence the lower precision, h_ε. h_ε can be viewed as the parameter determining the speed of consumers' inference about actual product quality or how noisy is the environment. One can see this from (3.5),

$$m_{t+1} = \frac{h_t m_t + h_\varepsilon z_t}{h_t + h_\varepsilon}$$

If the variance of the composite noise term is high, the level of precision, h_ε, is low. The posterior belief of the actual quality is the weighted sum of prior and the new signal about quality. High h_ε implies that the new signal is not very informative and consumers should put less weight on this when updating their belief. In the extreme case, where the variance of the composite noise approaches infinity or $h_\varepsilon \to 0$, the new signal is totally uninformative and consumers cannot learn from the new information at all, $m_{t+1} \to m_t$. The LHS of (3.11) is higher if h_ε increases. The firm will produce a higher quality product. In a less noisy environment, the firm will have more incentive to produce a high quality product because the consumer will refer high-perceived utility and previous consumers' report to a high product quality. In contrast, under a noisy environment, consumers will believe that a high-perceived utility and report are the consequence of noisy terms and not the firm's actual product quality. Knowing this, the firm will produce a lower quality product. In this simple model, the only incentive for the firm to produce a high quality product is to mislead consumers' belief about the true quality of the product.

The quality offered by the firm increases with the length of time that the product is in the market. As one can easily verify, the LHS of (3.11) equals 0 when T is 1 and it approaches $(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\left[(t-1)h_\varepsilon/h_t\right] > 0$ as $T \to \infty$. When T is small, consumers have less information about product quality and the product is discontinued early before the actual quality is precisely learnt; the firm thus misses a lot of revenue that it would get if the selling period had increased. This lowers the incentive for the firm to produce a higher quality product and to signal jam the learning process of consumers. The firm has a higher incentive to produce a better quality product if the relationship between product and consumers lasts for a longer period because high quality can mislead consumers into thinking that the product is a better match for their taste (the product has high γ) and will increase their willingness to pay.

4 Decision to Discontinue

The model described in Section 3 shows how product quality is determined and assumes that the product will be discontinued after T periods have elapsed. This section extends the model to the case where a firm introduces a new product after a previous one has been discontinued. Two simple examples are presented. Section 4.1 considers the case where the product discontinuation period is deterministic, i.e. a product lifespan is fixed. In Section 4.2, the product lifespan is variable but the section is restricted to the case where the firm has only one chance to discontinue the product. The more general case, where the discontinuation period is variable and there is an unrestricted number of discontinuations will be analysed in Section 5.

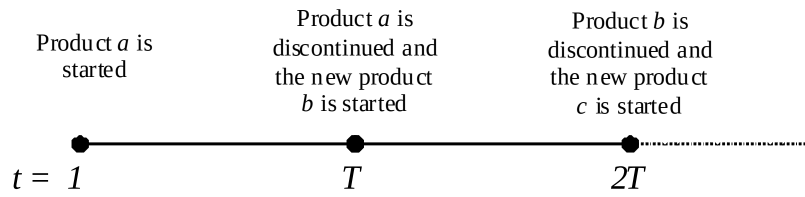

Figure 1: Deterministic Product Discontinuation Policy

4.1 Deterministic Product Discontinuation Period

It has been shown that the true product quality is revealed almost completely as time progresses. It is interesting to ask, if a firm chooses to discontinue an old product and introduce a new one, what would be the optimal time for discontinuation and what quality level would the firm select? With the opportunity to launch a new product, the firm faces two decision problems, the optimal quality problem and the product discontinuation problem. This section considers the case where selling periods are determined in advance and are the same for every product. There are many industries where firms decide not to wait until the actual quality is fully realised. They release a new design for their products every a fixed length of periods and consumers are forced to start the process of learning over again. For example, some Japanese and German car manufacturers have policies to release a new car design every 5 or 7 years. Fashion designers release their new designs every 3 months when the season changes. This section assumes that the firm discontinues its old product and launches its new one every T period; Figure 1 illustrates this. This deterministic product lifespan assumption will be relaxed in the subsequent sections where variable discontinuation periods are allowed.

Whenever a firm releases its new product, the quality choices must satisfy the FOC (3.11):

$$\frac{1-\delta}{1-\delta^T}\sum_{t=1}^{T}\delta^{t-1}\left[\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon+h_1}\right]=C'\left(q^*\right)$$

Since the selling period (T), the discount factor (δ) and the precision of the composite noise (h_ε) are the same for every product, the quality choice q^* for each product depends upon the precision of initial prior belief (h_1). As the idiosyncratic quality terms, γ, for all products produced by the firm are normally distributed with zero mean and identical precision, it is reasonable for the consumer to have initial prior beliefs with the same mean and precision every time a firm launches a new product. The following assumption states this formally.

Assumption 4.1 *Mean and precision of initial prior belief about quality noise at the time of launching are the same for every product i.e.,*

$$m_1 = m_{t_0}^b$$
$$h_1 = h_{t_0}^b$$

for all new product b launched at time t_0.

The following remark follows immediately from Assumption 4.1

Remark 4.2 *Under Assumption 4.1, optimal quality choices are identical for all products.*

In contract theory, γ can be viewed as a firm's type chosen randomly from a known distribution by nature. The quality choice chosen by a firm (an agent) can be viewed as its effort to signal jam and mislead consumer (or principal) learning of γ. In Holmström (1982), the manager's (agent's) ability or type is drawn by nature at the beginning and remains unchanged for all periods. In the limit, his ability will be fully known and the incentive for signal jamming will eventually disappear. The manager will exert less effort as his ability becomes more apparent. Holmström (1982)'s conclusion is that there exists no equilibrium in which a single firm repeatedly exerts high efforts. In contrast, the incentive to exert an effort in this model does not reduce as time progresses. If a firm chooses to launch a new product, quality noise for the new product will be drawn again from the same distribution function but is independent of the previous product's quality noise. Consumers have to start the whole process of learning again and the incentive for signal jamming for the firm and hence quality choice remains the same. In this model, a firm's type can change randomly by launching a new product. This high effort equilibrium is in line with what Mailath and Samuelson (1998) have emphasized; type changing is a device to obtain a sustained reputation effect. In fact, Holmström (1982) uses the same device by letting a firm's type follow a random walk process to obtain high effort equilibrium.

Let $y_t^a \equiv q_a^* + m_t^a - C(q_a^*)$ be the period t profit accrued to a firm from product a when the quality is chosen optimally at q_a^*. The total discounted return is

$$V = \sum_{i=1}^{T} \delta^{i-1} y_i^a + \delta^T \sum_{j=1}^{T} \delta^{j-1} y_j^b + \delta^{2T} \sum_{k=1}^{T} \delta^{k-1} y_k^c + ...$$

The expected return is

$$E[V] = E \sum_{i=1}^{T} \delta^{i-1} y_i^a + \delta^T E \left[\sum_{j=1}^{T} \delta^{j-1} y_j^b + \delta^T \sum_{k=1}^{T} \delta^{k-1} y_k^c + ... \right]$$

Since quality choices are identical for all products by Remark 4.2,

$$\begin{aligned} E[V] &= E \sum_{i=1}^{T} \delta^{i-1} y_i + \delta^T E[V] \\ &= \frac{\sum_{t=1}^{T} \delta^{t-1} E[y_t]}{1-\delta^T} \\ &= \frac{\sum_{t=1}^{T} \delta^{t-1} [q^*(T) + E(m_t) - C(q^*(T))]}{(1-\delta)\sum_{s=1}^{T} \delta^{s-1}} \end{aligned}$$

In equilibrium, $E(m_t) = m_1 = 0$. Then,

$$\begin{aligned} E[V] &= \frac{[q^*(T) - C(q^*(T))] \sum_{t=1}^{T} \delta^{t-1}}{(1-\delta)\sum_{s=1}^{T} \delta^{s-1}} \\ &= \frac{q^*(T) - C(q^*(T))}{(1-\delta)} \end{aligned} \tag{4.1}$$

The expected return (4.1) shows that the exogenously fixed product lifespan has no direct influence on the value function. However, it has an indirect influence through optimal quality choice. From the FOC for quality choice (3.11) and the discussion at the end of Section (3), the optimal quality increases with product lifespan. Proposition 4.3 shows that it is optimal for a firm not to discontinue its first product.

Proposition 4.3 *The optimal deterministic discontinuation period is to set $T \to \infty$.*

Proof. From (4.1), the firm's problem is to choose T that achieves the supremum of the expected return $E[V]$:

$$V^* = \sup_T \frac{q^*(T) - C(q^*(T))}{(1-\delta)}$$

The convex cost structure implies that the expected return is concave in q^* and maximised at $C'(q^*(T)) = 1$. From the FOC of quality choice (3.11),

$$\frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} \left(\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} \right) = C'(q^*(T))$$

the RHS, which is the marginal return from quality is increasing in T. Then,

$$\sup_T \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} \left(\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} \right) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \left(\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} \right)$$

But since $\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} \le 1$, then

$$(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \left(\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} \right) = \lim_{T\to\infty} C'(q^*(T)) \le 1$$

It has been shown that since marginal return from quality is increasing with T, when $T \to \infty$ the marginal return from quality is highest but still less than 1 and this gives a firm the highest possible expected return. ∎

This model of consumer learning suggests that a product with a longer selling period is of better quality. The reason for this is that the firm has a higher incentive to produce high quality product in order to bias consumer learning about the product quality noise term and increase consumer willingness to pay. However, this model also suggests that the firm should not discontinue the product and start a new one. According to the model, it is better to produce one product with a relatively long life than to produce many shorter life products. It seems that this simple model cannot explain why different firms choose different selling periods for their products. Incorporating demand or competition structures would enrich the present analysis. This issue is left for future research.

Note that extensions to the model, such as a fixed cost for introducing the new product or adding the waiting periods before the new product can be introduced, do not change above analysed results. The quality choice FOC (3.11) stays the same while the expected returns change slightly to

$$E[V] = \frac{[q^*(T) - C(q^*(T))] \sum_{t=1}^{T} \delta^{t-1} - F}{(1-\delta) \sum_{s=1}^{T+S} \delta^{s-1}}$$

where F is a fixed cost for launching the new product and S is the length of periods the firm has to wait before the new product can be launched after stopping the old one. These extensions do not affect quality choice but lower the expected returns.

4.2 When to Start a New Product? A Simple Illustration

In the previous section, the product discontinuation period is known and fixed in advance and the firm has to stop its product; even belief about product quality noise is high. This seems unreasonable. In general, the firm would be willing to stop its product when the product is believed to have low quality and consumers are willing to pay less for it. Therefore, the discontinuation period varies depending on belief about product quality and profit prospect given available information in each period. This section presents a simple illustration of the problems where the firm has only *one chance* to stop an old product and start a new one to show how the optimal product stopping time is determined and how this decision could or could not affect the quality choice. The structure of a more complicated decision problem where the firm can discontinue an old product and start a new one at any period and as many times as it wants will be discussed in Section 4.3. The optimal product discontinuation policy for this problem, which is one of the central questions of this paper, will be analysed in details in Section 5.

Suppose that the selling period is not fixed in advance, the quality choice chosen by the firm must satisfy the following FOC

$$(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \left(\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1} \right) = C'(q^*) \tag{4.2}$$

If a firm has only one chance to discontinue its product a and start the new product b, it will stop the product a at any time t only when expected profit from a conditional on information available in period t is lower than the expected profit from launching b. At time $t = t_0$, if firm continues to produce product a with quality q_a^*, it would get the expected profit of

$$
\begin{aligned}
\pi_a &= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[E\left[q_a^* + m_t \left| t = t_0 \right. \right] - C\left(q_a^* \right) \right] \\
&= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_a^* + E\left[m_t \left| t = t_0 \right. \right] - C\left(q_a^* \right) \right] \\
&= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_a^* + E\left[\frac{h_1 m_1 + h_\varepsilon \sum_{s=1}^{t-1} z_s}{h_t} \right] - C\left(q_a^* \right) \right] \\
&= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_a^* + \frac{h_1 m_1}{h_t} + \frac{h_\varepsilon}{h_t} \sum_{s=1}^{t_0-1} z_s + \frac{h_\varepsilon}{h_t} \sum_{s=t_0}^{t-1} E\left[z_s \left| t = t_0 \right. \right] - C\left(q_a^* \right) \right] \\
&= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_a^* + \frac{h_{t_0}}{h_t} \left[\frac{h_1 m_1}{h_{t_0}} + \frac{h_\varepsilon}{h_{t_0}} \sum_{s=1}^{t_0-1} z_s \right] + \frac{h_\varepsilon}{h_t} \sum_{s=t_0}^{t-1} m_{t_0} - C\left(q_a^* \right) \right] \\
&= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_a^* + \frac{h_{t_0} m_{t_0}}{(t-1)h_\varepsilon + h_1} + \frac{\left(t - t_0\right) h_\varepsilon}{(t-1)h_\varepsilon + h_1} m_{t_0} - C\left(q_a^* \right) \right] \\
&= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_a^* + \left[\frac{(t_0-1)h_\varepsilon + h_1}{(t-1)h_\varepsilon + h_1} + \frac{\left(t - t_0\right) h_\varepsilon}{(t-1)h_\varepsilon + h_1} \right] m_{t_0} - C\left(q_a^* \right) \right] \\
&= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_a^* + m_{t_0} - C\left(q_a^* \right) \right] \qquad (4.3)
\end{aligned}
$$

If, however, at $t = t_0$, it launches a new product b with quality q_b^*, the optimal quality must satisfy the FOC (4.2). Suppose the initial period prior belief of the idiosyncratic quality disturbance of product b, γ^b, has mean $m_{t_0}^b$ and precision $h_{t_0}^b$. Optimal quality must satisfy the following FOC:

$$
\begin{aligned}
(1-\delta) \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[\frac{(t-t_0)h_\varepsilon}{h_t} \right] &= C'\left(q_b^* \right) \\
(1-\delta) \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[\frac{(t-t_0)h_\varepsilon}{\left(t - t_0\right) h_\varepsilon + h_{t_0}^b} \right] &= C'\left(q_b^* \right) \qquad (4.4)
\end{aligned}
$$

Comparing (4.4) with (4.2), one can see that q_b^* will be higher than q_a^* if the initial prior belief of idiosyncratic quality disturbance of b has higher precision. Or, $q_a^* \lesseqgtr q_b^* \leftrightarrow h_1 \gtreqless h_{t_0}^b$. However, if $h_1 = h_{t_0}^b$ as assumed in Assumption 4.1, quality choices are identical.

As stated in Remark 4.2, the firm chooses the same level of quality when launching a new product because all parameters are the same every time it makes the decision.

The firm will launch a new product in the period when the prior belief falls below some critical level. The following proposition states this.

Proposition 4.4 *There exists a unique m^* such that a new product is launched at period t_0 if and only if $m_{t_0} < m^*$ and if all new products have the same initial prior belief as stated in Assumption 4.1, $m^* = m_1$.*

Proof. Profit from launching a new product with quality q_b^* at time t_0 is

$$\begin{aligned}\pi_b &= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[E\left(q_b^* + m_t^b \right) - C\left(q_b^*\right) \right] \\ &= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_b^* + \frac{h_{t_0}^b m_{t_0}^b}{h_t} + \frac{h_\varepsilon}{h_t} \sum_{s=t_0}^{t-1} \left(m_{t_0}^b + q_b^* - q_b^* \right) - C\left(q_b^*\right) \right] \\ &= \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_b^* + \frac{h_{t_0}^b m_{t_0}^b}{(t-t_0)\, h_\varepsilon + h_{t_0}^b} + \frac{(t-t_0) h_\varepsilon}{(t-t_0)\, h_\varepsilon + h_{t_0}^b} m_{t_0}^b - C\left(q_b^*\right) \right] \end{aligned} \tag{4.5}$$

At time $t = t_0$, a firm will stop old product a and launch new product b if $\pi_a \leq \pi_b$. Or from (4.3) and (4.5), one can show that

$$\begin{aligned} &\sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_a^* + m_{t_0} - C\left(q_a^*\right) \right] \\ &\qquad \leq \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[q_b^* + \frac{h_{t_0}^b m_{t_0}^b}{(t-t_0)\, h_\varepsilon + h_{t_0}^b} + \frac{(t-t_0) h_\varepsilon}{(t-t_0)\, h_\varepsilon + h_{t_0}^b} m_{t_0}^b - C\left(q_b^*\right) \right]. \end{aligned}$$

By Remark 4.2, the above inequality can be simplified to

$$\sum_{t=t_0}^{\infty} \delta^{t-t_0} m_{t_0} \leq \sum_{t=t_0}^{\infty} \delta^{t-t_0} \left[\frac{h_{t_0}^b m_{t_0}^b}{(t-t_0)\, h_\varepsilon + h_{t_0}^b} + \frac{(t-t_0) h_\varepsilon}{(t-t_0)\, h_\varepsilon + h_{t_0}^b} m_{t_0}^b \right] \tag{4.6}$$

The LHS term of (4.6) is linear and increasing in m_{t_0} while the RHS term is constant in m_{t_0}. Then, there exists a unique $m_{t_0}^*$ that satisfies (4.6) with equality making firm indifferent from continuing old product or launching the new one. This proves the first part of Proposition 4.4.

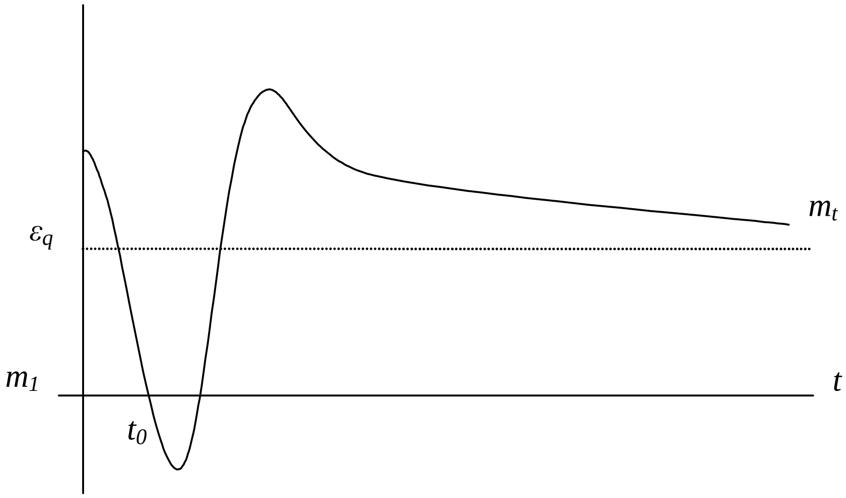

Figure 2: Time path of m_t and product stopping at t_0

Assumption 4.1 implies

$$\sum_{t=t_0}^{\infty} \delta^{t-t_0} m_{t_0} \leq \sum_{t=t_0}^{\infty} \delta^{t-t_0} m_1. \tag{4.7}$$

If all new products launched by the firm have the same initial prior belief about the quality disturbance term, the critical level m^* that makes the firm indifferent from launching a new or continuing with an old product is m_1. This proves the second part of the proposition. ■

This means that the firm will choose to stop the old product and produce the new one immediately when prior belief about quality noise of the present period falls below the initial prior belief when a product is newly introduced. Figure 2 illustrates one of the possible time paths of m_t.

From the figure, if the firm cannot stop selling the product, the prior mean of a quality noise will converge to the true quality noise almost surely. However, if the firm can choose to discontinue the old product, it will choose to stop in the period t_0 when m_{t_0} falls below the initial prior mean about the quality noise of a newly launched product, m_1. It is not necessary that a good quality product will survive. A high quality product can be judged as a product with lower than standard quality and can be discontinued after a series of unlucky events during the initial phases of its introduction.

In this simple case, the quality choice is not affected by the discontinuation decision because the firm has only one chance to launch the new product. In a more enriched setting where the firm can choose to stop its product at any time, the firm has to take into account the probability that the product will be discontinued under the optimal discontinuation rule when choosing quality. In addition, as will be shown in Section 5.4, optimal quality choice determines the expected profit from launching the new product and affects the optimal critical level m^* for which a product will be discontinued.

4.3 Structure of the Decision Problems

The quality choice and product stopping rule obtained in Section 4.2 are actually based on the assumption that a firm has only one chance of stopping the old product a and launching the new product b. After b has been introduced to the market, the firm will produce b forever. The structure of the problem is changed significantly if in every period the firm can decide whether to discontinue the old product and start the new one. Note that Assumption 4.1 will still be applied here and Remark 4.2 is still valid in this setting because the future looks the same every time the firm chooses to start the new product. However, this implies that the optimal stopping policy would be identical for every product as well.

Remark 4.5 *Under Assumption 4.1, optimal product discontinuation policies are identical for all products.*

The firm has two levels of decision; it starts by choosing quality choice, and then enters the product discontinuation game. Firstly, it has to choose a level of quality to maximise its expected profit from the product and q is the decision variable. Since a product could be discontinued by product discontinuation policy, the profitability of a product depends upon the policy implemented in the product discontinuation game. Then, quality choice, q, must be chosen optimally conditional on optimal product discontinuation policy being implemented. Similarly, the optimal discontinuation policy must be determined given the optimal level of quality, $q = q^*$. Therefore, solutions to the two decision problems will be determined simultaneously.

5 Product Discontinuation Policies

This section will show that optimal product stopping policies telling a firm when to launch a new product and maximises an expected discounted profit exist. The analysis starts by explaining the structure of the problem in Section 5.1. Section 5.2 shows that the rather complicated discontinuation problem involving stopping and restarting products can be re-expressed or transformed to a simple or typical stopping problem which is easier to work with. Then, Section 5.3 shows the existence of optimal product stopping policy to this problem. In Section 5.4, the optimal product stopping policy is derived and some comparative static results (holding an optimal quality choice constant) are presented in Section 5.5.

5.1 Structure of the Product Discontinuation Problem

A firm in this model has to make two decisions, when to optimally stop an old product and start a new one, and what would be the optimal quality choice for the new product. Given that quality is chosen optimally, in each period after observing m_t, a firm can choose whether to continue with the present product or launch the new one. The strategy of the firm is given by

$$\psi : \mathbb{R} \longrightarrow \{0, 1\}.$$

In this decision rule, 0 indicates a stoppage and 1 indicates the decision to continue the old product. The optimal product stopping policy consists of the sequence of these functions

$$\mathbf{\Psi} = (\psi_1, \psi_2, ...).$$

that maximises the total expected discounted return. It is assumed that the new product can be launched as many times as required and that the same stopping policy is used for each product. For any product a and at any time t, having observed a sequence $\{m_1^a, m_2^a, ..., m_t^a\}$ and received $\sum_{s=1}^{t} \delta^{s-1} [q_a^* + m_s^a - C(q_a^*)]$, the firm may want to discontinue product a and start with product b over again. Since the same stopping policy is used in every product and period and a sequence $\{m_1^a, m_2^a, ..., m_t^a\}$ is independent for each product, the optimal product lifespan for each product T_a,T_b, ... is i.i.d. random variables. As in Section 4.1, let

$$y_t^a \equiv q_a^* + m_t^a - C(q_a^*)$$

be the period t profit accrued to firm from product a. For notation simplicity, superscript a may be dropped if it is not required in the context. The total discounted return is

$$V = \sum_{i=1}^{T_a} \delta^{i-1} y_i^a + \delta^{T_a} \sum_{j=1}^{T_b} \delta^{j-1} y_j^b + \delta^{T_a+T_b} \sum_{k=1}^{T_c} \delta^{k-1} y_k^c + ...$$

The expected return is

$$\begin{aligned} E[V] &= E\sum_{i=1}^{T_a} \delta^{i-1} y_i^a + E\delta^{T_a} E\left[\sum_{j=1}^{T_b} \delta^{j-1} y_j^b + \delta^{T_b} \sum_{k=1}^{T_c} \delta^{k-1} y_k^c + ...\right] && (5.1) \\ &= E\sum_{i=1}^{T_a} \delta^{i-1} y_i^a + E\delta^{T_a} E[V] \\ &= \frac{E\sum_{t=1}^{T} \delta^{t-1} y_t}{1 - E\delta^T} \\ E[V] &= \frac{E\sum_{t=1}^{T} \delta^{t-1} y_t}{(1-\delta)\, E\sum_{t=1}^{T} \delta^{t-1}} && (5.2) \end{aligned}$$

The optimal product stopping policy is the rule that maximises $E[V]$. Let $\Im$ denote the class of product discontinuation rules,

$$\Im = \left\{ T : T \geq 1, 0 < \sum_{i=1}^{T} \delta^{i-1} < \infty \right\} \tag{5.3}$$

The objective is to find a discontinuation policy $T \in \Im$ to achieve the supremum in

$$V^* = \sup_{T \in \Im} \frac{E\sum_{t=1}^{T} \delta^{t-1} y_t}{(1-\delta)\, E\sum_{t=1}^{T} \delta^{t-1}} \tag{5.4}$$

Two conditions are required for V^* to be well defined. First, an optimal product stopping policy must exist. This issue will be addressed in Section 5.3. Second, in order to avoid the problem of dealing with $0/0$ or ∞/∞, the following condition is required.

$$0 < \sum_{i=1}^{T} \delta^{i-1} < \infty \tag{5.5}$$

The left inequality of (5.5) is satisfied because the firm is assumed to sell the new product

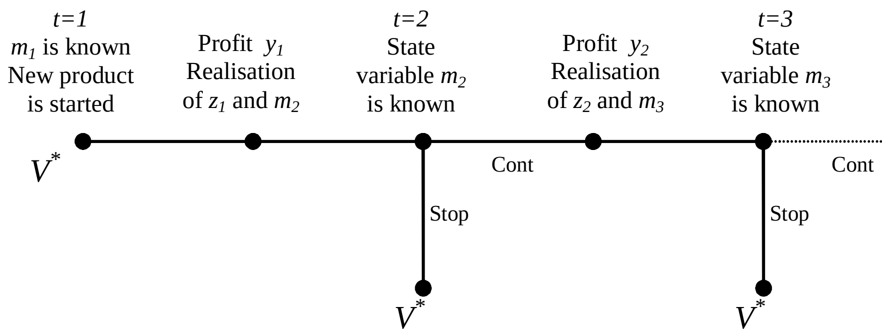

Figure 3: Structure of the stopping problem

in the first period and observe at least one observation of z_t. The left inequality is also satisfied since the discount factor $0 < \delta < 1$ by assumption in Section 3. The structure of this stopping problem for any arbitrary product is shown in Figure 3. From the figure, at the start of each period t, a prior mean of product quality noise is known and the firm has to make a decision to stop or continue the product conditional on this state variable. If the firm chooses to continue the same product, it will receive state profit y_t and the value of random variable z_t is realised. Knowing z_t, the prior mean of product quality noise is updated to m_{t+1} and the firm enters period $t + 1$. Alternatively, if the firm chooses to stop the present product and start a new one, the whole process of learning will start all over again. If optimal discontinuation policy is employed, the firm will receive the total expected profit of V^* from this decision.

5.2 Original and Transformed Problems

This section illustrates how one can transform the product discontinuation problem (5.4) to a simpler problem and use the transformed problem to find an optimal product discontinuation policy and the value function of the original one. Section 5.3 shows that optimal product discontinuation policies for transformed and hence original problems exist.

The purpose of this section is to show that the solution of the problem presented in

Section 5.1 can be obtained from the solution of the following problem

$$\overline{V} = \sup_{T \in \Im} E\left(\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right) \tag{5.6}$$

for some λ. The following lemma shows that the two problems are related.

Lemma 5.1 *If for some* λ, $\sup_{T \in \Im} E\left(\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right) = 0$, *then*

$$\sup_{T \in \Im} \frac{E\sum_{t=1}^{T}\delta^{t-1}y_t}{\left(1-\delta\right)E\sum_{t=1}^{T}\delta^{t-1}} = \lambda$$

Proof. If $\sup_{T \in \Im} E\left(\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right) = 0$ then for every stopping rule T, $E\left(\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right) \leq 0$. It follows that

$$E\left(\sum_{t=1}^{T}\delta^{t-1}y_t - \lambda\left(1-\delta\right)\sum_{t=1}^{T}\delta^{t-1}\right) \leq 0$$

$$\frac{E\sum_{t=1}^{T}\delta^{t-1}y_t}{\left(1-\delta\right)E\sum_{t=1}^{T}\delta^{t-1}} \leq \lambda$$

and that

$$\sup_{T \in \Im} \frac{E\sum_{t=1}^{T}\delta^{t-1}y_t}{\left(1-\delta\right)E\sum_{t=1}^{T}\delta^{t-1}} = \lambda$$

■

The next proposition claims that any solution for (5.6) is also optimal for the original problem.

Proposition 5.2 *If* $\sup_{T \in \Im} E\left(\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right) = 0$ *is attained at* T^*, *then* T^* *is optimal for maximising* $\sup_{T \in \Im} \frac{E\sum_{t=1}^{T}\delta^{t-1}y_i}{(1-\delta)E\sum_{t=1}^{T}\delta^{t-1}}$.

Proof. Suppose not, from Lemma 5.1, $\frac{E\sum_{t=1}^{T}\delta^{t-1}y_i}{(1-\delta)E\sum_{t=1}^{T}\delta^{t-1}} < \lambda$ and then

$$E\left(\sum_{t=1}^{T^*}\delta^{t-1}y_t - \lambda\left(1-\delta\right)\sum_{t=1}^{T^*}\delta^{t-1}\right) < 0$$
$$E\left(\sum_{t=1}^{T^*}\delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right) < 0$$

but this contradicts the fact that T^* is optimal and achieves the supremum in $\sup_{T\in\Im} E\left(\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right) = 0$ ■

Proposition 5.2 allows the transformation of the original maximisation problem (5.4) to the ordinary stopping problem (5.6) with the return when choosing to stop at time T being $\sum_{i=1}^{T}\delta^{i-1}\left[y_i-\lambda\left(1-\delta\right)\right]$. If the optimal stopping rule for a transformed problem exists, then it is also optimal for the product stopping problem. From now on, the problem (5.4) will be referred to as the *original problem* and (5.6) as the *transformed problem*. Therefore, V^* and $\overline{V}$ are the value function of the original and transformed problems respectively.

5.3 Existence of Optimal Product Discontinuation Policies

The objective of this section is to prove that the optimal product discontinuation policy for the problem described in Section 5.1 exists so that V^* expressed in (5.4) can be attained. It is shown in Lemma 5.1 from the previous section that the original product discontinuation problem can be transformed to problem (5.6) with an appropriate value of λ. In addition, by Proposition 5.2, if the optimal discontinuation policy for the transformed problem exists, then so does the original problem. The task now is to show that optimal stopping rules for problem (5.6) exist. The following lemmas show that the transformed product stopping problem possesses two basic properties that are sufficient for an optimal stopping rule to exist.

Lemma 5.3 $E\left[\sup_{T\in\Im}\left(\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right)\right] < \infty$

Proof.

$$\sup_{T\in\Im}\left(\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right) = \sum_{t=1}^{T}\delta^{t-1}\left[q^*+m_t-C\left(q^*\right)-\lambda\left(1-\delta\right)\right]$$

where $m_t = \frac{h_1 m_1 + h_\varepsilon \sum_{s=1}^{t-1} z_s}{h_1+(t-1)h_\varepsilon}$ and $m_t\,|m_1$ is normally distributed with a zero mean and variance $\sigma^2_{m_t} = (t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon+h_1}\right]^2 \frac{h_1+h_\varepsilon}{h_1 h_\varepsilon}$. (see Lemma 5.13 on page 40 below)

Therefore,

$$\begin{aligned}
E\left[\sup_{T\in\Im}\left(\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right)\right] &= \sum_{t=1}^{T}\delta^{t-1}\left[q^*+E\left(m_t\right)-C\left(q^*\right)-\lambda\left(1-\delta\right)\right] \\
&= \sum_{t=1}^{T}\delta^{t-1}\left[q^*-C\left(q^*\right)-\lambda\left(1-\delta\right)\right] \\
&= \left[q^*-C\left(q^*\right)-\lambda\left(1-\delta\right)\right]\frac{1-\delta^T}{1-\delta} \\
&< \infty
\end{aligned}$$

■

Lemma 5.4 $\limsup_{T\to\infty}\left(\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right) \le \sum_{t=1}^{\infty}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]$ *almost surely.*

Proof. Since $\lim_{T\to\infty}\left(\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right) = \sum_{t=1}^{\infty}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]$ and, for a real-valued sequence, $\lim_{T\to\infty} s_T = s$ implies $\limsup_{T\to\infty} s_T = \liminf_{T\to\infty} s_T = s$. It follows that $\limsup_{T\to\infty}\left(\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right) = \sum_{t=1}^{\infty}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]$ almost surely and the lemma is proved. ■

Lemma 5.3 makes it possible to interchange summation and expectation in subsequent arguments of this section. It can be shown that Lemma 5.3 and Lemma 5.4 are sufficient for the existence of optimal stopping rules. The method of Chow and Robbins (1963) and Ferguson (2000) using the notion of regular stopping rules is applied to show the existence of optimal stopping rules. The following definition defines the concept of regular stopping rules introduced by Snell (1952).

Definition 5.5 *A discontinuation policy T is said to be regular, if for every s,*

$$E\left[\left.\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right| m_1,...,m_s\right] > \sum_{t=1}^{s}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]$$

almost surely on $\{T>s\}$.

From Definition 5.5, stopping rule T is regular if it tells a firm to continue at time s and gets a higher expected return than from stopping at that state. The following lemma shows that for any stopping rule there is a regular one that gives at least as good an expected payoff.

Lemma 5.6 *Given any discontinuation policy T, there is a regular discontinuation policy T' such that*

$$E\left[\sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] \geq E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right].$$

Proof. Define

$$T^{'} = \min\left\{s \geq 0 : E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\middle| m_1, ..., m_s\right] \leq \sum_{t=1}^{s} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right\} \tag{5.7}$$

That is, T' tells the firm to continue to use T and stop when T tells it to continue, but stop is at least as good. It is clear that T' is a discontinuation policy and that $T' \leq T$. On $\{T' = s\}$, $E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\middle| m_1, ..., m_s\right] \leq \sum_{t=1}^{s} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]$ almost surely for all s, while on $\{T' = \infty\}$, it is true that $\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right] = \sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right] = \sum_{t=1}^{\infty} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]$ almost surely. Hence, for any $1 \leq T' < \infty$,

$$\begin{aligned} &E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] \\ &\qquad = E\left[\sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] + E\left[\sum_{t=T'+1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] \end{aligned} \tag{5.8}$$

Consider the last term of (5.8), $E\left[\sum_{t=T'+1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right]$, since $T' \leq T$, the analysis can be separated into 2 cases.

Case I If $T' = T$, the last term of (5.8) disappears and then,

$$E\left[\sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] = E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right]$$

Case II If $T' < T$, according to the discontinuation policy defined in (5.7),

$$E\left[\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_{T'}\right]-\sum_{t=1}^{T'}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\leq 0$$

$$\sum_{t=1}^{T}E\left(\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_{T'}\right)-\sum_{t=1}^{T'}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\leq 0$$

$$\sum_{t=T'+1}^{T}E\left(\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_{T'}\right)\leq 0$$

$$E\left[\sum_{t=T'+1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right]\leq 0$$

and from (5.8), it follows that

$$E\left[\sum_{t=1}^{T'}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right]\geq E\left[\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right]$$

The interchange between summation and expectation is valid by Lemma 5.3.

It remains to prove that T' is regular. Note that on $\{T' > s\}$,

$$E\left[\sum_{t=1}^{T'}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_s\right]\geq E\left[\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_s\right]$$

almost surely. Since

$$E\left[\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_s\right]>\sum_{t=1}^{s}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]$$

almost surely on $\{T' > s\}$, then

$$E\left[\sum_{t=1}^{T'}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_s\right]>\sum_{t=1}^{s}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]$$

almost surely on $\{T' > s\}$. ■

The next lemma shows that, among regular optimal product discontinuation policies, the one that tells the firm to stop later yields no worse return.

Lemma 5.7 *If T and T' are regular product discontinuation policies, then so is $T'' = \max\{T, T'\}$ and then*

$$E\left[\sum_{t=1}^{T''} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] \geq \max\left\{E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right], E\left[\sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right]\right\}$$

Proof. If $T' = T$, $T'' = \max\{T', T\} = T' = T$, then

$$\begin{aligned} E\left[\sum_{t=1}^{T''} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] &= E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] \\ &= E\left[\sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] \end{aligned}$$

If $T' \neq T$, assuming without loss of generality that $T' > T$, so $T'' = \max\{T', T\} = T'$. Since T' is regular and $T' > T$,

$$E\left[\sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right] \middle| m_1, ..., m_T\right] > \sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]$$

almost surely. By the law of iterated expectation,

$$E\left[\sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] > E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right]$$

and since $T'' = T'$,

$$\begin{aligned} E\left[\sum_{t=1}^{T''} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] &= E\left[\sum_{t=1}^{T'} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] \\ &> E\left[\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right] \end{aligned}$$

It remains to show that T'' is regular. Assuming again without loss of generality that $T' > T$ so that $T'' = T'$ and from above result

$$E\left[\sum_{t=1}^{T''}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right]\geq E\left[\sum_{t=1}^{T'}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right]$$

From regularity of T', it can be shown that

$$\begin{aligned}E\left[\sum_{t=1}^{T''}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_s\right] &\geq E\left[\sum_{t=1}^{T'}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\middle| m_1,...,m_s\right]\\ &> \sum_{t=1}^{s}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\end{aligned}$$

almost surely on $\{T'' = T' > s\}$. This implies that T'' is regular. ■

By Lemma 5.6, the attention can be restricted just on regular product discontinuation policies. In addition, Lemma 5.7 shows that the set of all regular discontinuation policies is an ordered set. In other words, Lemma 5.7 enables one to rank between any pair of regular discontinuation policies. With these two lemmas, it is unable to show the existence of optimal product discontinuation policies for the firm. The following proposition, which shows the existence of optimal product discontinuation policy, is based on Ferguson (2000). The proposition is the application of Fatou Lemma which states that suppose that the sequence of functions is a sequence of random variables $X_{1,}X_{2,...}$ with $X_n \leq Y$ almost surely for some Y such that $E\left(|Y|\right) < \infty$, then $\limsup_{n\to\infty} EX_n \leq E\limsup_{n\to\infty} X_n$.

Proposition 5.8 *There exists a product discontinuation policy T^* such that*

$$E\left[\sum_{t=1}^{T^*}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right]=\overline{V}$$

where $\overline{V} = \sup_{T\in\Im} E\left[\sum_{t=1}^{T}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right]$.

Proof. Let $\{T_1, T_2, \ldots\}$ be any sequence of stopping rule such that

$$\lim_{j\to\infty} E\left[\sum_{t=1}^{T_j}\delta^{t-1}\left[y_t-\lambda\left(1-\delta\right)\right]\right]=\overline{V}$$

From Lemma 5.6, There is always a sequence of regular rules $\{T_1', T_2', \ldots\}$ such that $\lim_{j\to\infty} E\left[\sum_{t=1}^{T_j'} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] = \overline{V}$. Let $T_j'' = \max\left\{T_1', T_2', \ldots, T_j'\right\}$, then by Lemma 5.7

$$E\left[\sum_{t=1}^{T_j''} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] \geq E\left[\sum_{t=1}^{T_j'} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right]$$

which implies that $\lim_{j\to\infty} E\left[\sum_{t=1}^{T_j''} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] = \overline{V}$. It follows that

$$\overline{V} = \limsup_{j\to\infty} E\left[\sum_{t=1}^{T_j''} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] \tag{5.9}$$

and define $T^* = \sup\{T_1', T_2', \ldots\}$. Note that T_j'' is a nondecreasing sequence of stopping policies converging to stopping rule T^*. Therefore, from Lemma 5.4, $\limsup_{j\to\infty} \sum_{t=1}^{T_j''} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right] \leq \sum_{t=1}^{T^*} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]$ almost surely. Taking expectation to both sides of the inequality yields

$$E\left[\limsup_{j\to\infty} \sum_{t=1}^{T_j''} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] \leq E\left[\sum_{t=1}^{T^*} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] \tag{5.10}$$

Using Fatou Lemma, one can relate (5.9) and (5.10),

$$\begin{aligned}\overline{V} = \limsup_{j\to\infty} E\left[\sum_{t=1}^{T_j''} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] &\leq E\left[\limsup_{j\to\infty} \sum_{t=1}^{T_j''} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] \\ &\leq E\left[\sum_{t=1}^{T^*} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right]\end{aligned}$$

But since $E\left[\sum_{t=1}^{T^*} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] \leq \overline{V}$ by the definition of $\overline{V}$,

$$E\left[\sum_{t=1}^{T^*} \delta^{t-1}\left[y_t - \lambda\left(1-\delta\right)\right]\right] = \overline{V}$$

and the proposition is proved. ■

5.4 Optimal Product Discontinuation Policy

It was presented in the previous section that optimal discontinuation policies exist. The objective of this section is to derive the optimal product stopping policy that maximises the firm's expected total profit (5.1). Note that the structure of the product stopping problem described in Section 5.1 differs from ordinary stopping problems in two aspects. First, a firm can always launch a new product immediately after it chooses to stop the old product. Therefore, the new stopping problem is restarted again after the new product is launched and optimal stopping rules must consider this. This is different from ordinary stopping problems where the process ends after the decision maker chooses to stop. However, it was shown in Lemma 5.1 and Proposition 5.2 that the original restart problem (5.4) can be transformed to an ordinary stopping problem (5.6) with an appropriate value λ. Second, at any time t after selling product and gaining period profit of $y_t = q^* + m_t - C(q^*)$, random variable z_t is realised and the posterior distribution of quality noise is updated. The posterior mean, m_{t+1}, and hence period $t+1$ profit, $y_{t+1} = q^* + m_{t+1} - C(q^*)$, are known to the firm at the beginning of period $t+1$ before the decision to stop or continue is made. This contrasts to ordinary stopping problems where payoffs from continuing are realised only after the decision to continue is made.

The objective for the original restart problem is to choose an optimal stopping policy, T^*, such that it maximises the expected total profit (5.1) or equivalently (5.2) which is rewritten here,

$$E[V] = \frac{E\sum_{t=1}^{T}\delta^{t-1}y_t}{(1-\delta)E\sum_{t=1}^{T}\delta^{t-1}}$$

The numerator term of the RHS of (5.2) is the expected discounted profit from one product accrued to the firm by using stopping rule T. The term $E\sum_{i=1}^{T}\delta^{i-1}$ in the denominator can be viewed as an expected discounted time used for one product to obtain the expected profit in the numerator. Since a firm can always launch a new product, the problem is an infinite horizon problem and $(1-\delta)$ is multiplied to express the denominator term in normalised or *per period* form. Therefore, $E[V]$ can be viewed not only as the expected discounted total profit over an infinite horizon but also as the expected average profit of firm per effective unit of time or the expected *rate of return*. It is important to note that the objective is, in effect, to choose an optimal stopping rule that maximises average profit or expected rate of return and *not* the expected discounted profit from a product.

As discussed above, Lemma 5.1 enables the transformation of original problem (5.4),

$$V^* = \sup_{T \in \Im} \frac{E \sum_{t=1}^{T} \delta^{t-1} y_t}{(1-\delta) E \sum_{t=1}^{T} \delta^{t-1}}$$

to an ordinary stopping problem of the form in (5.6), $\sup_{T\in\Im} E\left(\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda(1-\delta)\right]\right)$, for some value of λ. This transformed problem can be rewritten as

$$\sup_{T\in\Im}\left[E\sum_{t=1}^{T}\delta^{t-1}y_t - \lambda(1-\delta)E\sum_{t=1}^{T}\delta^{t-1}\right]$$

and λ in this transformed problem can be explained as a shadow value of an effective unit of time measured in the same unit as the payoffs, and is as if the firm is being charged λ for each effective unit of time. In addition, according to Lemma 5.1 and Proposition 5.2, if λ is chosen at $\lambda = \lambda^*$ such that $\sup_{T\in\Im} E\left(\sum_{t=1}^{T} \delta^{t-1}\left[y_t - \lambda^*(1-\delta)\right]\right) = 0$, the optimal expected total profit for the firm, V^*, will also equal to λ^*. In this case, λ^* can be viewed as the maximised expected total return over an infinite horizon that the firm can get from launching the new product using the optimal stopping policy.

To derive an optimal product discontinuation policy for a firm, the method given by Lemma 5.1 and Proposition 5.2 will be utilised. The strategy is to solve the transformed problem for an optimal stopping rule and the associated value function and choose λ^* so that the value function of the transformed problem is equal to zero. Then, the value function of the original problem, V^*, is equal to λ^*.

The next proposition shows that the optimal stopping rule for the transformed problem is to stop at the first t for which $m_t \le m^*(\lambda, q^*)$ for some λ.

Proposition 5.9 *Optimal product discontinuation policy T^* for the transformed problem (5.6), $\sup_{T\in\Im} E\left(\sum_{i=1}^{T} \delta^{i-1}\left[y_i - \lambda(1-\delta)\right]\right)$, is in the form*

$$T^* = \min\left\{t > 1 : m_t \le m^*(\lambda, q^*)\right\}$$

where $m^*(\lambda, q^*) = \lambda(1-\delta) - [q^* - C(q^*)]$

Proof. Suppose at time t, $\sum_{i=1}^{t-1} \delta^{i-1}\left[y_i - \lambda(1-\delta)\right] = Y^*$ and it is optimal to stop. By the principal of optimality, $\sum_{i=1}^{t-1} \delta^{i-1}\left[y_i - \lambda(1-\delta)\right]$ is at least as large as any future return $E\left[\sum_{i=1}^{T} \delta^{i-1}\left[y_i - \lambda(1-\delta)\right]\right]$ for all stopping rules T or

$\sum_{i=1}^{t-1} \delta^{i-1} \left[y_i - \lambda(1-\delta)\right] \geq E\left[\sum_{i=1}^{T} \delta^{i-1} \left[y_i - \lambda(1-\delta)\right]\right]$. Then,

$$\sum_{i=1}^{t-1} \delta^{i-1} \left[y_i - \lambda(1-\delta)\right] \geq \sum_{i=1}^{t-1} \delta^{i-1} \left[y_i - \lambda(1-\delta)\right] + E\left[\sum_{i=t}^{T} \delta^{i-1} \left[y_i - \lambda(1-\delta)\right]\right]$$

$$0 \geq \sum_{i=t}^{T} \delta^{i-1} \left[q^* + E(m_i) - C(q^*) - \lambda(1-\delta)\right]$$

$$0 \geq \left[q^* + m_t - C(q^*) - \lambda(1-\delta)\right] \left(\delta^{t-1} + \cdots + \delta^{T-1}\right)$$

But since $\left(\delta^{t-1} + \cdots + \delta^{T-1}\right) > 0$, then

$$0 \geq \left[q^* + m_t - C(q^*) - \lambda(1-\delta)\right]$$

$$\lambda(1-\delta) - \left[q^* - C(q^*)\right] \geq m_t$$

$$m^*(\lambda, q^*) \geq m_t$$

Then, it is optimal to stop only when $m_t \leq m^*(\lambda, q^*)$ and the optimal rule T^* must be of the form $T^* = \min\{t > 1 : m_t \leq m^*(\lambda, q^*)\}$. ■

The following corollary states the optimal product discontinuation policy for the original problem.

Corollary 5.10 *Under optimal product discontinuation policy, a firm continues a product in period t if profit from continuation in period t is at least as large as the average optimal profit, or otherwise discontinues.*

Proof. Let λ^* be the chosen value of λ such that the value function of the transformed problem is maximised at 0, that is

$$\overline{V} = \sup_{T \in \Im} E\left[\sum_{i=1}^{T} \delta^{i-1} \left[y_i - \lambda^*(1-\delta)\right]\right] = 0$$

From Proposition 5.9, the optimal stopping rule for the transformed problem is $T^* = \min\{t > 1 : m_t \leq m^*(\lambda^*, q^*)\}$. However, from Proposition 5.2, T^* is also the optimal product discontinuation policy for the firm and this achieves the supremum in (5.4),

$$V^* = \sup_{T \in \Im} \frac{E \sum_{i=1}^{T} \delta^{i-1} y_i}{(1-\delta) E \sum_{i=1}^{T} \delta^{i-1}}$$

Since λ^* is chosen such that $\overline{V} = 0$, by Lemma 5.1, $V^* = \lambda^*$. Therefore, under the optimal product discontinuation policy, the firm will stop the old product and launch the new one at period t if

$$\begin{aligned}
m_t &\leq m^* \left(\lambda^*, q^*\right) \\
m_t &\leq \lambda^* \left(1 - \delta\right) - \left[q^* - C\left(q^*\right)\right] \\
q^* + m_t - C\left(q^*\right) &\leq \lambda^* \left(1 - \delta\right) \\
q^* + m_t - C\left(q^*\right) &\leq \left(1 - \delta\right) V^*
\end{aligned} \tag{5.11}$$

The LHS term of (5.11) is the firm's profit from continuing the product in period t. Since V^* is the optimal expected discounted total profit over the infinite horizon under the optimal discontinuation policy, the term $(1 - \delta) V^*$ in the RHS of (5.11) is an average or per period return under the implementations of the optimal discontinuation policy. ■

In other words, Corollary 5.10 shows that the optimal product discontinuation policy is in the following form

$$\begin{aligned}
T^* &= \min \left\{t > 1 : y_t \leq \lambda^* \left(1 - \delta\right)\right\} \\
&= \min \left\{t > 1 : m_t \leq \lambda^* \left(1 - \delta\right) - \left[q^* - C\left(q^*\right)\right]\right\}
\end{aligned}$$

In still other words, the optimal product discontinuation policy, T^*, suggests the firm stops the first t when $m_t \leq m^* \left(\lambda^*, q^*\right)$ or $m_t \leq \lambda^* \left(1 - \delta\right) - \left[q^* - C\left(q^*\right)\right]$ where λ^* is the optimal expected discounted profit over the infinite horizon. Since m_t depend upon the realisation of random variables z_t, m_t are random variables. The optimal product discontinuation period is, then, a random variable and the firm can conjecture the probability that its current product will be continued at any given period t under the implementation of the optimal discontinuation policy. In fact, for any given discontinuation policy $T = \min \left\{t > 1 : m_t \leq m^* \left(\lambda, q^*\right)\right\}$ and product quality choice q, the firm can calculate the probability that its newly launched product will survive in any period t. Definition 5.11 defines this probability.

Definition 5.11 *For any product discontinuation policy* $T = \min \left\{t > 1 : m_t \leq m^* \left(\lambda, q^*\right)\right\}$ *and any quality choice* q*, let* κ_t *be the probability that the current product will be continued* *in period t under the implementation of T. Then,*

$$\kappa_t(q,\lambda) = P\left[\bigcap_{j\leq t}(m_j \geq m^*(\lambda, q^*))\right]$$

A product will be continued in period t only when it survives the discontinuation policy in every previous period and κ_t denotes the probability of this survival. κ_t can be expressed in terms of distribution functions of random variables γ and ε_s for $s = 1, ..., t$. To show this, κ_2, κ_3 and κ_4 will be derived. The expression of κ_t for any $t > 1$ will be subsequently derived in Lemma 5.12.

At the beginning of period $t = 2$, the market has realised random variable z_1 from period 1 and updated the prior belief of γ, $m_2 = \frac{h_1 m_1 + h_\varepsilon z_1}{h_1 + h_\varepsilon}$. Therefore, the probability that the firm will continue the product in period 2 is

$$\begin{aligned}\kappa_2(q,\lambda) &= P\left[m_2 \geq m^*(\lambda, q^*)\right] = P\left[\frac{h_1 m_1 + h_\varepsilon z_1}{h_1 + h_\varepsilon} \geq m^*(\lambda, q^*)\right] \\ &= P\left[z_1 \geq m^*(\lambda, q^*) \cdot \left(\frac{h_1 + h_\varepsilon}{h_\varepsilon}\right)\right]\end{aligned}$$

Recalling from (3.3) that $z_t = \gamma + q + \varepsilon_t - q^*$ and defining $\widetilde{m}_2 = m^*(\lambda, q^*) \cdot \left(\frac{h_1 + h_\varepsilon}{h_\varepsilon}\right)$,

$$\begin{aligned}\kappa_2(q,\lambda) &= P\left[\gamma + q - q^* + \varepsilon_1 \geq \widetilde{m}_2\right] \\ &= P\left[\gamma + \varepsilon_1 \geq \widetilde{m}_2 - (q - q^*)\right] \\ &= P\left[\varepsilon_1 \geq \widetilde{m}_2 - (\gamma + (q - q^*))\right] \\ &= \int_{-\infty}^{\infty}\int_{\widetilde{m}_2 - (\gamma + q - q^*)}^{\infty} f(\gamma, \varepsilon_1)\, d\varepsilon_1 d\gamma\end{aligned}$$

by independence and normality assumptions,

$$\begin{aligned}\kappa_2(q,\lambda) &= \int_{-\infty}^{\infty}\int_{\widetilde{m}_2 - (\gamma + q - q^*)}^{\infty} f(\gamma) . f(\varepsilon_1)\, d\varepsilon_1 d\gamma \\ &= \int_{-\infty}^{\infty}\int_{\widetilde{m}_2 - (\gamma + q - q^*)}^{\infty} \frac{1}{(2\pi)\,\sigma_q \sigma_\varepsilon} \exp\left\{-\left(\frac{\gamma^2}{2\sigma_q^2}\right) - \left(\frac{\varepsilon_1^2}{2\sigma_\varepsilon^2}\right)\right\} d\varepsilon_1 d\gamma\end{aligned}$$

where $f(\cdot)$ is the distribution function of interested variables, i.e.,

$$f(\gamma) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_q} \exp\left\{-\frac{\gamma^2}{2\sigma_q^2}\right\}$$

and

$$f(\varepsilon_1) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_\varepsilon} \exp\left\{-\frac{\varepsilon_1^2}{2\sigma_e^2}\right\}.$$

For $t = 3$, the probability of old product continuation is

$$\begin{aligned}
\kappa_3(q,\lambda) &= P\left[m_3 \geq m^*(\lambda, q^*) \cap m_2 \geq m^*(\lambda, q^*)\right] \\
&= P\left[\frac{h_1 m_1 + h_\varepsilon(z_1 + z_2)}{h_1 + 2h_\varepsilon} \geq m^*(\lambda, q^*) \cap z_1 \geq m^*(\lambda, q^*) \cdot \left(\frac{h_1 + h_\varepsilon}{h_\varepsilon}\right)\right] \\
&= P\left[z_1 + z_2 \geq m^*(\lambda, q^*) \cdot \left(\frac{h_1 + 2h_\varepsilon}{h_\varepsilon}\right) \cap z_1 \geq m^*(\lambda, q^*) \cdot \left(\frac{h_1 + h_\varepsilon}{h_\varepsilon}\right)\right] \\
&= P\left[z_1 + z_2 \geq \widetilde{m}_3 \cap z_1 \geq \widetilde{m}_2\right]
\end{aligned}$$

where $\widetilde{m}_j = m^*(\lambda, q^*) \cdot \left(\frac{h_1 + (j-1)h_\varepsilon}{h_\varepsilon}\right)$. Since $z_t = \gamma + q + \varepsilon_t - q^*$,

$$\begin{aligned}
\kappa_3(q,\lambda) &= P\left[2(\gamma + q - q^*) + \varepsilon_1 + \varepsilon_2 \geq \widetilde{m}_3 \cap \gamma + q - q^* + \varepsilon_1 \geq \widetilde{m}_2\right] \\
&= P\left[\varepsilon_2 \geq \widetilde{m}_3 - (2(\gamma + q - q^*) + \varepsilon_1) \cap \varepsilon_1 \geq \widetilde{m}_2 - (\gamma + q - q^*)\right] \\
&= \int_{-\infty}^{\infty} \int_{\widetilde{m}_2 - (\gamma + q - q^*)}^{\infty} \int_{\widetilde{m}_3 - 2(\gamma + q - q^*) - \varepsilon_1}^{\infty} f(\gamma, \varepsilon_1, \varepsilon_2)\, d\varepsilon_2 d\varepsilon_1 d\gamma
\end{aligned}$$

by independence and normality assumptions,

$$\begin{aligned}
f(\gamma, \varepsilon_1, \varepsilon_2) &= f(\gamma) . f(\varepsilon_1) . f(\varepsilon_2) \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_q \sigma_\varepsilon^2} \exp\left\{-\left(\frac{\gamma^2}{2\sigma_q^2}\right) - \left(\frac{\varepsilon_1^2 + \varepsilon_2^2}{2\sigma_\varepsilon^2}\right)\right\}
\end{aligned}$$

where $f(\cdot)$ is the distribution function of interested variables, i.e.,

$$f(\gamma) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_q} \exp\left\{-\frac{\gamma^2}{2\sigma_q^2}\right\}$$

and

$$f\left(\varepsilon_s\right)=\frac{1}{(2\pi)^{\frac{1}{2}}\,\sigma_{\varepsilon}}\exp\left\{-\frac{\varepsilon_s^2}{2\sigma_{\varepsilon}^2}\right\}.$$

for $s=1,2$.

For $t=4$, the probability of old product continuation is

$$\begin{aligned}
\kappa_4\left(q,\lambda\right)&=P\left[m_4\geq m^*\left(\lambda,q^*\right)\cap m_3\geq m^*\left(\lambda,q^*\right)\cap m_2\geq m^*\left(\lambda,q^*\right)\right]\\
&=P\left[\frac{h_1m_1+h_{\varepsilon}(z_1+z_2+z_3)}{h_1+3h_{\varepsilon}}\geq m^*\left(\lambda,q^*\right)\cap m_3\geq m^*\left(\lambda,q^*\right)\cap m_2\geq m^*\left(\lambda,q^*\right)\right]\\
&=P\left[\begin{array}{l} z_1+z_2+z_3\geq m^*\left(\lambda,q^*\right)\left(\frac{h_1+3h_{\varepsilon}}{h_{\varepsilon}}\right)\cap z_2+z_1\geq m^*\left(\lambda,q^*\right)\left(\frac{h_1+2h_{\varepsilon}}{h_{\varepsilon}}\right)\\ \cap z_1\geq m^*\left(\lambda,q^*\right)\left(\frac{h_1+h_{\varepsilon}}{h_{\varepsilon}}\right)\end{array}\right]
\end{aligned}$$

Since, again, $\widetilde{m}_j=m^*\left(\lambda,q^*\right)\cdot\left(\frac{h_1+(j-1)h_{\varepsilon}}{h_{\varepsilon}}\right)$ and $z_t=\gamma+q+\varepsilon_t-q^*$

$$\begin{aligned}
\kappa_4\left(q,\lambda\right)&=P\left[z_1+z_2+z_3\geq\widetilde{m}_4\cap z_2+z_1\geq\widetilde{m}_3\cap z_1\geq\widetilde{m}_2\right]\\
&=P\left[\bigcap_{j=1}^{4}\left(\sum_{s=1}^{j-1}z_s\geq\widetilde{m}_j\right)\right]\\
&=P\left[\bigcap_{j=1}^{4}\left(\sum_{s=1}^{j-1}\left(\gamma+q-q^*+\varepsilon_s\right)\geq\widetilde{m}_j\right)\right]\\
&=P\left[\bigcap_{j=1}^{4}\left(\left(j-1\right)\left(\gamma+q-q^*\right)+\sum_{s=1}^{j-1}\varepsilon_s\geq\widetilde{m}_j\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\kappa_4\left(q,\lambda\right)=P\left[\bigcap_{j=1}^{4}\left(\varepsilon_{j-1}\geq\widetilde{m}_j-\left(j-1\right)\left(\gamma+q-q^*\right)-\sum_{s=1}^{j-2}\varepsilon_s\right)\right]\\
&=\int_{-\infty}^{\infty}\int_{\widetilde{m}_2-(\gamma+q-q^*)}^{\infty}\int_{\widetilde{m}_3-2(\gamma+q-q^*)-\varepsilon_1}^{\infty}\int_{\widetilde{m}_4-3(\gamma+q-q^*)-(\varepsilon_1+\varepsilon_2)}^{\infty}f\left(\gamma,\varepsilon_1,\varepsilon_2,\varepsilon_3\right)d\varepsilon_3d\varepsilon_2d\varepsilon_1d\gamma
\end{aligned}$$

and $f(\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the joint probability distribution function and

$$\begin{aligned} f(\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3) &= f(\gamma) f(\varepsilon_1) f(\varepsilon_2) f(\varepsilon_3) \\ &= \frac{1}{(2\pi)^{\frac{4}{2}} \sigma_q \sigma_\varepsilon^3} \exp\left\{ -\left(\frac{\gamma^2}{2\sigma_q^2}\right) - \left(\frac{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}{2\sigma_\varepsilon^2}\right) \right\} \end{aligned}$$

by independence and normality assumptions.

Lemma 5.12 shows the expression of κ_t in terms of γ and ε_s for any $t > s \geq 1$.

Lemma 5.12 *For any product discontinuation policy* $T = \min\{t > 1 : m_t \leq m^*(\lambda, q^*)\}$ *and any quality choice* q*, the probability that the current product will be continued in period* t *under the implementation of* T *is*

$$\kappa_t(q, \lambda) = \int_{-\infty}^{\infty} \int_{\widetilde{m}_2 - (\gamma + q - q^*)}^{\infty} \cdots \int_{\widetilde{m}_t - (t-1)(\gamma + q - q^*) - \sum_{s=1}^{t-2} \varepsilon_s}^{\infty} f(\gamma) f(\varepsilon_1) \cdots f(\varepsilon_{t-1})\, d\varepsilon_{t-1} \ldots d\varepsilon_1 d\gamma \tag{5.12}$$

where $\widetilde{m}_j = m^*(\lambda, q^*) \cdot \left(\frac{h_1 + (j-1) h_\varepsilon}{h_\varepsilon}\right)$,

$$f(\gamma) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_q} \exp\left\{ -\frac{\gamma^2}{2\sigma_q^2} \right\}$$

$$f(\varepsilon_s) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_\varepsilon} \exp\left\{ -\frac{\varepsilon_s^2}{2\sigma_\varepsilon^2} \right\}$$

for $s \in \{1, 2, \ldots t-1\}$ *and*

$$f(\gamma) f(\varepsilon_1) \cdots f(\varepsilon_{t-1}) = \frac{1}{(2\pi)^{\frac{t}{2}} \sigma_q \sigma_\varepsilon^{t-1}} \exp\left\{ -\left(\frac{\gamma^2}{2\sigma_q^2}\right) - \left(\frac{\varepsilon_1^2 + \cdots + \varepsilon_{t-1}^2}{2\sigma_\varepsilon^2}\right) \right\}$$

Proof. From Definition 5.11,

$$\begin{aligned}\kappa_t(q,\lambda) &= P\left[\bigcap_{j\le t}(m_j \ge m^*(\lambda,q^*))\right]\\ &= P\left[\bigcap_{j\le t}\left(\frac{h_1 m_1 + h_\varepsilon\sum_{s=1}^{j-1} z_s}{h_1+(j-1)h_\varepsilon} \ge m^*(\lambda,q^*)\right)\right]\end{aligned}$$

since $m_1 = 0$ and $\widetilde{m}_j = m^*(\lambda,q^*)\cdot\left(\frac{h_1+(j-1)h_\varepsilon}{h_\varepsilon}\right)$,

$$\begin{aligned}\kappa_t(q,\lambda) &= P\left[\bigcap_{j\le t}\left(\sum_{s=1}^{j-1} z_s \ge m^*(\lambda,q^*)\cdot\left(\frac{h_1+(j-1)h_\varepsilon}{h_\varepsilon}\right)\right)\right]\\ &= P\left[\bigcap_{j\le t}\left(\sum_{s=1}^{j-1} z_s \ge \widetilde{m}_j\right)\right]\end{aligned}$$

Recalling from (3.3) that $z_t = \gamma + q + \varepsilon_t - q^*$,

$$\begin{aligned}\kappa_t(q,\lambda) &= P\left[\bigcap_{j\le t}\left(\sum_{s=1}^{j-1}(\gamma+q-q^*+\varepsilon_s) \ge \widetilde{m}_j\right)\right]\\ &= P\left[\bigcap_{j\le t}\left((j-1)(\gamma+q-q^*)+\sum_{s=1}^{j-1}\varepsilon_s \ge \widetilde{m}_j\right)\right]\\ &= P\left[\bigcap_{j\le t}\left(\sum_{s=1}^{j-1}\varepsilon_s \ge \widetilde{m}_j-(j-1)(\gamma+q-q^*)\right)\right]\\ &= P\left[\bigcap_{j\le t}\left(\varepsilon_{j-1} \ge \widetilde{m}_j-(j-1)(\gamma+q-q^*)-\sum_{s=1}^{j-2}\varepsilon_s\right)\right]\\ &= \int_{-\infty}^{\infty}\int_{\widetilde{m}_2-(\gamma+q-q^*)}^{\infty}\cdots\int_{\widetilde{m}_t-(t-1)(\gamma+q-q^*)-\sum_{s=1}^{t-2}\varepsilon_s}^{\infty} f(\gamma,\varepsilon_1,\ldots,\varepsilon_{t-1})\,d\varepsilon_{t-1}\ldots d\varepsilon_1 d\gamma\end{aligned}$$

where $f(\gamma, \varepsilon_1, \ldots, \varepsilon_{t-1})$ is the joint probability distribution function and

$$\begin{aligned} f(\gamma, \varepsilon_1, \ldots, \varepsilon_{t-1}) &= f(\gamma) f(\varepsilon_1) \cdots f(\varepsilon_{t-1}) \\ &= \frac{1}{(2\pi)^{\frac{t}{2}} \sigma_q \sigma_\varepsilon^{t-1}} \exp\left\{ -\left(\frac{\gamma^2}{2\sigma_q^2}\right) - \left(\frac{\varepsilon_1^2 + \cdots + \varepsilon_{t-1}^2}{2\sigma_\varepsilon^2}\right)\right\} \end{aligned}$$

by independence and normality assumptions. ■

Having found the optimal discontinuation policy, $T^* = \min\{t > 1 : m_t \leq m^*(\lambda^*, q^*)\}$ $= \min\{t > 1 : m_t \leq \lambda^*(1-\delta) - [q^* - C(q^*)]\}$, the remaining problem is to find the value of λ^*. However, as Lemma 5.1 suggested, λ^* is the value of λ that makes the optimal value function for the transformed problem equal to zero or,

$$\overline{V} = \sup_{T \in \Im} E\left[\sum_{t=1}^{T} \delta^{t-1} [y_t - \lambda^*(1-\delta)]\right] = 0$$

It is important to show some results that will facilitate subsequent analysis. Some of the following are basic and standard results from the theory of probability.

Lemma 5.13 *Conditional on the information at the time of launching new product, for any $t > 1$, quality choice q, and any $a \in \mathbb{R}$,*

$$m_t \left| m_1, q \sim N\left(\mu_t(q), \sigma_{m_t}^2\right)\right.$$

and at equilibrium quality, $q = q^*$,

$$m_t \left| m_1 \sim N\left(0, \sigma_{m_t}^2\right)\right.$$

where $\mu_t(q) = \frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1}(q - q^*)$ and $\sigma_{m_t}^2 = (t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right]^2 \frac{h_1 + h_\varepsilon}{h_1 h_\varepsilon}$

Proof. Conditional on information at time $t = 1$, the distribution of random variables $z_s = \gamma + q + \varepsilon_s - q^*$ for any s is normal with mean

$$E[z_s | t = 1] = E[\gamma | t = 1] + q - q^* = m_1 + q - q^*$$

but since initial prior belief $m_1 = 0$, $E[z_s | t = 1] = q - q^*$. For the variance, since γ and

ε_s are independent, variance of z_s is

$$\begin{aligned} var\left(z_s\left|t=1\right.\right) &= var\left(\gamma\left|t=1\right.\right) + var\left(\varepsilon_s\left|t=1\right.\right) \\ &= \frac{1}{h_1} + \frac{1}{h_\varepsilon} \\ &= \frac{h_1 + h_\varepsilon}{h_1 h_\varepsilon} \end{aligned}$$

That is, $z_s \sim N\left(q - q^*, \frac{h_1+h_\varepsilon}{h_1 h_\varepsilon}\right)$.

Since $m_t = \frac{h_1 m_1 + h_\varepsilon \sum_{s=1}^{t-1} z_s}{(t-1)h_\varepsilon + h_1}$, m_t is a linear combination of normal distributed random variables z_s. Therefore, conditional on information at time $t = 1$, for any $t > 1$ m_t is normally distributed with mean:

$$\begin{aligned} E\left[m_t\left|t=1\right.\right] &= \frac{h_\varepsilon \sum_{s=1}^{t-1} E\left[z_s\left|t=1\right.\right]}{(t-1)h_\varepsilon + h_1} \\ &= \frac{(t-1)\,h_\varepsilon}{(t-1)h_\varepsilon + h_1}\left(q - q^*\right) \end{aligned}$$

and variance:

$$var\left[m_t\left|t=1\right.\right] = \left[\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right]^2 var\left[\sum_{s=1}^{t-1} z_s\left|t=1\right.\right]$$

since z_s are independent

$$\begin{aligned} var\left[m_t\left|t=1\right.\right] &= \left[\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right]^2 \sum_{s=1}^{t-1} var\left[z_s\left|t=1\right.\right] \\ &= (t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right]^2 \frac{h_1 + h_\varepsilon}{h_1 h_\varepsilon} \end{aligned}$$

Therefore, $m_t|\, m_1, q \sim N\left(\mu_t\left(q\right), \sigma_{m_t}^2\right)$ where $\mu_t\left(q\right) = \frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1}\left(q - q^*\right)$, $\sigma_{m_t}^2 = (t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon+h_1}\right]^2 \frac{h_1+h_\varepsilon}{h_1 h_\varepsilon}$.

At equilibrium quality choice, $q^* = q$, $\mu_t\left(q\right) = 0$ for all t and the distribution of m_t conditional on the information at time of launching at equilibrium quality is $m_t\left|m_1\right. \sim N\left(0, \sigma_{m_t}^2\right)$ ■

Lemma 5.14 *Conditional on the information at the time of launching the new product,* *for any $t > 1$, quality choice q, and any $a \in \mathbb{R}$*

$$P\left[m_t > a\right] = 1 - \Phi\left(\frac{a - \mu_t\left(q\right)}{\sigma_{m_t}}\right)$$

and at equilibrium quality, $q = q^*$

$$P\left[m_t > a\right] = 1 - \Phi\left(\frac{a}{\sigma_{m_t}}\right)$$

where $\mu_t\left(q\right) = \frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1}\left(q - q^*\right)$, $\sigma_{m_t}^2 = (t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right]^2 \frac{h_1 + h_\varepsilon}{h_1 h_\varepsilon}$ and $\Phi\left(\cdot\right)$ is the standard normal cdf.

Proof. From Lemma 5.13, $m_t \left| m_1, q \sim N\left(\mu_t\left(q\right), \sigma_{m_t}^2\right)\right.$. Let $m_t = \mu_t\left(q\right) + \sigma_{m_t} v$ where $v \sim N\left(0, 1\right)$, then

$$\begin{aligned} P\left[m_t > a\right] &= P\left[\mu_t\left(q\right) + \sigma_{m_t} v > a\right] \\ &= P\left[v > \frac{a - \mu_t\left(q\right)}{\sigma_{m_t}}\right] \\ &= 1 - \Phi\left(\frac{a - \mu_t\left(q\right)}{\sigma_{m_t}}\right) \end{aligned}$$

At equilibrium quality choice, $q^* = q$, $\mu_t\left(q\right) = 0$ for all t

$$P\left[m_t > a\right] = 1 - \Phi\left(\frac{a}{\sigma_{m_t}}\right)$$

where $\Phi\left(\cdot\right)$ is the standard normal cdf. ■

Lemma 5.15 (Inverse Mills Ratio) *Conditional on the information at the time of launching the new product, for any $t > 1$, quality choice q, and any $a \in \mathbb{R}$*

$$E\left[m_t \middle| m_t \geq a\right] = \mu_t\left(q\right) + \sigma_{m_t} \frac{\phi\left(\frac{a - \mu_t(q)}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{a - \mu_t(q)}{\sigma_{m_t}}\right)}$$

and at equilibrium quality, $q = q^*$

$$E\left[m_t\right| m_t \geq a] = \sigma_{m_t} \frac{\phi\left(\frac{a}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{a}{\sigma_{m_t}}\right)}$$

where $\mu_t(q) = \frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1}(q-q^*)$, $\sigma^2_{m_t} = (t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon+h_1}\right]^2 \frac{h_1+h_\varepsilon}{h_1 h_\varepsilon}$, $\phi(\cdot)$ *and* $\Phi(\cdot)$ *are the standard normal pdf. and cdf. respectively.*

Proof. Let $m_t = \mu_t(q) + \sigma_{m_t} v$ where $v \sim N(0,1)$ and $\mu_t(q) = \frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1}(q-q^*)$, then

$$\begin{aligned} E\left[m_t\right| m_t \geq a] &= \int_a^\infty m_t \cdot f\left(m_t \left|m_t \geq a\right.\right) dm_t \\ &= \int_{\frac{a-\mu_t(q)}{\sigma_{m_t}}}^\infty \left(\mu_t(q) + \sigma_{m_t} v\right) \frac{1}{\sigma_{m_t}} \frac{\phi(v)}{1-\Phi\left(\frac{a-\mu_t(q)}{\sigma_{m_t}}\right)} d\sigma_{m_t} v \\ &= \frac{1}{1-\Phi\left(\frac{a-\mu_t(q)}{\sigma_{m_t}}\right)} \left[\mu_t(q) \int_{\frac{a-\mu_t(q)}{\sigma_{m_t}}}^\infty \phi(v)\, dv + \sigma_{m_t} \int_{\frac{a-\mu_t(q)}{\sigma_{m_t}}}^\infty v\phi(v)\, dv\right] \end{aligned}$$

Since, for standard normal distribution, $\phi'(v) = -v\phi(v)$,

$$\begin{aligned} E\left[m_t\right| m_t \geq a] &= \frac{1}{1-\Phi\left(\frac{a-\mu_t(q)}{\sigma_{m_t}}\right)} \left[\mu_t(q) \int_{\frac{a-\mu_t(q)}{\sigma_{m_t}}}^\infty \phi(v)\, dv - \sigma_{m_t} \int_{\frac{a-\mu_t(q)}{\sigma_{m_t}}}^\infty \phi'(v)\, dv\right] \\ &= \frac{1}{1-\Phi\left(\frac{a-\mu_t(q)}{\sigma_{m_t}}\right)} \left[\mu_t(q)\left(1-\Phi\left(\frac{a-\mu_t(q)}{\sigma_{m_t}}\right)\right) + \sigma_{m_t}\phi\left(\frac{a-\mu_t(q)}{\sigma_{m_t}}\right)\right] \\ &= \mu_t(q) + \sigma_{m_t} \frac{\phi\left(\frac{a-\mu_t(q)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{a-\mu_t(q)}{\sigma_{m_t}}\right)} \end{aligned}$$

At equilibrium quality choice, $q^* = q$, $\mu_t(q^*) = 0$ for all t and

$$E\left[m_t\right| m_t \geq a] = \sigma_{m_t} \frac{\phi\left(\frac{a}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{a}{\sigma_{m_t}}\right)}$$

■

Recalling from Proposition 5.9 that the optimal product discontinuation policy is in the form $T^* = \min\{t > 1 : m_t \leq m^*(\lambda, q^*)\}$ where $m^*(\lambda, q^*) = \lambda(1-\delta) - [q^* - C(q^*)]$ and it achieves the supremum of $\overline{V}(\lambda) = \sup_{T \in \Im} E\left[\sum_{t=1}^{T} \delta^{t-1}[y_t - \lambda(1-\delta)]\right]$, the value function of the transformed problem (5.6). One can find the value of $\overline{V}(\lambda)$ by considering the special case where the firm implements T^* and T^* tells the firm to continue the product forever. That is the case when $m_t \geq m^*(\lambda, q^*)$ for every $t \in \mathbb{N}$. The expected profit from this case must be $\overline{V}(\lambda)$ because this is the case when the firm implements T^* forever and, therefore, acts optimally in every period. Proposition 5.16 derives $\overline{V}(\lambda)$ using this argument.

Proposition 5.16 *Let $T^* = \min\{t > 1 : m_t \leq m^*(\lambda, q^*)\}$ be the optimal product discontinuation policy. The value function of the transformed problem (5.6) can be expressed in the following form*

$$\overline{V}(\lambda) = \sum_{t=1}^{\infty} \kappa_t(\lambda)\,\delta^{t-1}\left[E\left(m_t | \, m_t \geq m^*(\lambda, q^*)\right) - m^*(\lambda, q^*)\right]$$

or

$$\overline{V}(\lambda) = \sum_{t=1}^{\infty} \kappa_t(\lambda)\,\delta^{t-1}\left(q^* + \sigma_{m_t}\frac{\phi\left(\frac{m^*(\lambda,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda,q^*)}{\sigma_{m_t}}\right)} - C(q^*) - \lambda(1-\delta)\right) \qquad (5.13)$$

where $m^*(\lambda, q^*) = \lambda(1-\delta) - [q^* - C(q^*)]$, $\sigma^2_{m_t} = (t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right]^2 \frac{h_1 + h_\varepsilon}{h_1 h_\varepsilon}$ and $\kappa_t(\lambda)$ is $\kappa_t(q, \lambda)$ from Lemma 5.12 evaluated at $q = q^*$.

Proof. From the definition of value function

$$\overline{V}(\lambda) = \sup_{T \in \Im} E\left(\sum_{i=1}^{T} \delta^{i-1}[y_i - \lambda(1-\delta)]\right)$$

Since T^* is the optimal product discontinuation policy,

$$\overline{V}(\lambda) = E\left[\sum_{i=1}^{T^*} \delta^{i-1}[y_i - \lambda(1-\delta)] \middle| m_i \geq m^*(\lambda, q^*)\right]$$

Consider the case when $m_t \geq m^*(\lambda, q^*)$ for every $t \in \mathbb{N}$ and the policy T^* calls a firm to continue forever. The value function, $\overline{V}(\lambda)$, is equal to the expected discounted profit over the infinite horizon,

$$\overline{V}(\lambda)=\sum_{t=1}^{\infty}\kappa_t(\lambda)\,\delta^{t-1}E\left[q^*+m_t-C(q^*)-\lambda(1-\delta)|\,m_t\geq m^*(\lambda,q^*)\right]$$

where $\kappa_t(\lambda)=\int_{-\infty}^{\infty}\int_{\widetilde{m}_2-\gamma}^{\infty}\cdots\int_{\widetilde{m}_t-(t-1)\gamma-\sum_{s=1}^{t-2}\varepsilon_s}^{\infty}f(\gamma)\,f(\varepsilon_1)\cdots f(\varepsilon_{t-1})\,d\varepsilon_{t-1}\ldots d\varepsilon_1 d\gamma$ is the probability that a firm will continue the product in period t derived by Lemma 5.12 where quality choice q has been set optimally at q^*. Since all terms in the bracket of the above expression are constant except random variables m_t, it can be rewritten as

$$\overline{V}(\lambda)=\sum_{t=1}^{\infty}\kappa_t(\lambda)\,\delta^{t-1}\left[q^*+E\left(m_t|\,m_t\geq m^*(\lambda,q^*)\right)-C(q^*)-\lambda(1-\delta)\right]$$

By Lemma 5.13, $m_t|\,m_1\sim N\left(0,\sigma_{m_t}^2\right)$ where $\sigma_{m_t}^2=(t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon+h_1}\right]^2\frac{h_1+h_\varepsilon}{h_1h_\varepsilon}$, and Lemma 5.15, it follows that

$$\overline{V}(\lambda)=\sum_{t=1}^{\infty}\kappa_t(\lambda)\,\delta^{t-1}\left(q^*+\sigma_{m_t}\frac{\phi\left(\frac{m^*(\lambda,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda,q^*)}{\sigma_{m_t}}\right)}-C(q^*)-\lambda(1-\delta)\right)$$

and this proves the proposition. ■

Corollary 5.17 concludes the optimal product discontinuation policy and the value function for the original product discontinuation problem (5.4).

Corollary 5.17 *The optimal product discontinuation policy and the value function of the original problem are, respectively, in the following forms,*

$$T^*=\min\{t>1:m_t\leq m^*(\lambda^*,q^*)\}=\min\{t>1:m_t\leq\lambda^*(1-\delta)-[q^*-C(q^*)]\}$$

and

$$V^*=\sup_{T\in\Im}\frac{E\sum_{t=1}^{T}\delta^{t-1}y_t}{(1-\delta)\,E\sum_{t=1}^{T}\delta^{t-1}}=\frac{E\sum_{t=1}^{T^*}\delta^{t-1}y_t}{(1-\delta)\,E\sum_{t=1}^{T^*}\delta^{t-1}}=\lambda^*$$

where λ^* satisfies the following equation

$$\sum_{t=1}^{\infty}\kappa_t(\lambda^*)\,\delta^{t-1}\left[E\left(m_t|\,m_t\geq m^*(\lambda^*,q^*)\right)-m^*(\lambda^*,q^*)\right]=0 \tag{5.14}$$

or

$$\sum_{t=1}^{\infty} \kappa_t (\lambda^*) \delta^{t-1} \left(q^* + \sigma_{m_t} \frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)} - C(q^*) - \lambda^*(1-\delta) \right) = 0 \tag{5.15}$$

Proof. The proof is a straightforward application of Lemma 5.1 and Proposition 5.2 and hence omitted. ■

5.5 Comparative Static Analysis

This section analyses how the firm would change its optimal discontinuation policy when exogenous variables such as the precision of initial belief (h_1), the informativeness of the noisy signal (h_ε) and the discount factor (δ) vary. It is important to note that the equilibrium optimal discontinuation policy and the equilibrium quality, considered in Section 6, are determined simultaneously. The comparative static results for both endogenous variables should be analysed together as there is a feedback link between them. However, for simplicity, the quality choice will be treated as a constant and the first order effect of variables mentioned above to the product discontinuation decision will be considered.

The condition determining the optimal discontinuation policy is (5.14) shown in Corollary 5.17, which can be rewritten here as (the arguments of $m^*(\lambda^*, q^*)$ and $\kappa_t(\lambda^*)$ are dropped and the terms are rewritten m^* and κ_t in all the below expressions to make the notations compact)

$$\sum_{t=1}^{\infty} \kappa_t \delta^{t-1} \left[E\left(m_t | m_t \geq m^*\right) - m^* \right] = \sum_{t=1}^{\infty} \kappa_t \delta^{t-1} \left(\sigma_{m_t} \frac{\phi\left(\frac{m^*}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*}{\sigma_{m_t}}\right)} - m^* \right) = 0 \tag{5.16}$$

Equation (5.16) where the Implicit Function Theorem will be applied is central for the comparative static analysis. Some important expressions are rewritten here as

$$\sigma_{m_t}^2 = (t-1)\left(\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right)^2 \frac{h_1 + h_\varepsilon}{h_1 h_\varepsilon}$$

$$\kappa_t = \int_{-\infty}^{\infty} \int_{\widetilde{m}_2-\gamma}^{\infty} \cdots \int_{\widetilde{m}_t-(t-1)\gamma-\sum_{s=1}^{t-2}\varepsilon_s}^{\infty} f(\gamma) f(\varepsilon_1) \cdots f(\varepsilon_{t-1})\, d\varepsilon_{t-1} \ldots d\varepsilon_1 d\gamma$$

$$\widetilde{m}_t = m^* \cdot \left(\frac{h_1 + (t-1)\, h_\varepsilon}{h_\varepsilon} \right)$$

and

$$E\left(m_t \middle| \, m_t \geq m^*\right) = \sigma_{m_t} \frac{\phi\left(\frac{m^*}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*}{\sigma_{m_t}}\right)}$$

Terms κ_t and $E\left(m_t \middle| \, m_t \geq m^*\right) - m^*$ in (5.14) are the functions of the optimal stopping policy (m^*), the precision of the quality disturbance term (h_1), and the precision of composite noise (h_ε). It is important to understand how κ_t and $E\left(m_t \middle| \, m_t \geq m^*\right) - m^*$ change as m^*, h_1 and h_ε change since this will be useful for the subsequent comparative static analysis. For κ_t, an increase in m^* raises $\widetilde{m}_t$ for all t. A higher $\widetilde{m}_t$ increases all the lower limits of integration of variable ε_t in the κ_t expression. This means that $\frac{\partial \kappa_t}{\partial \widetilde{m}} < 0$ for all t. Therefore, the probability that a product will survive the discontinuation policy decreases. The same analysis can be carried out for h_1 and h_ε by examining how they affect $\widetilde{m}_t$. Remark 5.18 restates these results

Remark 5.18 *The probability that the product will be continued in period t, κ_t,* ***decreases*** *as*

1. *m^* increases or $\frac{\partial \kappa_t}{\partial m^*} < 0$ for all $m^* \in \mathbb{R}$,*
2. *h_1 increases or $\frac{\partial \kappa_t}{\partial h_1} < 0$ for all $h_1 \in \mathbb{R}^+$ and*
3. *h_ε decreases or $\frac{\partial \kappa_t}{\partial h_\varepsilon} > 0$ for all $h_\varepsilon \in \mathbb{R}^+$.*

Remark 5.18.1 is simple and intuitive; if the policy for which the product will be continued is rigid, then the chances that it will survive such rule after the realisation of m_t are slim. For Remark 5.18.2, a higher h_1 implies that the market puts more weight on initial prior belief m_1 when forming prior belief m_t for future periods. As a result, m_t tends to lie closer to $m_1 = 0$ and the product is less likely to survive the discontinuation policy $m^* > 0$. It is an opposite story when h_ε is high. As h_ε gets higher, the market places more weight in the realisation of the new information z_t relative to initial prior belief m_1. If h_ε is high or the environment is less noisy, the market would attribute the high realised value of z_t to the high level of quality disturbance term, γ, and it is more likely that z_t will be high for the future period t as well. Therefore, if the product can survive for some periods it is more likely that for each period t, the prior belief m_t will exceed m^* and the product survives the discontinuation policy.

For $E\left(m_t \middle| m_t \geq m^*\right) - m^*$, this term is always greater than 0 and one can verify that because m_t has zero mean, $\left[E\left(m_t \middle| m_t \geq m^*\right) - m^*\right] \rightarrow \infty$ as $m^* \rightarrow -\infty$ and $\left[E\left(m_t \middle| m_t \geq m^*\right) - m^*\right] \rightarrow 0$ as $m^* \rightarrow \infty$. Since m_t, in the perspective of the firm (see Lemma 5.13), is normally distributed and one can plot $\left[E\left(m_t \middle| m_t \geq m^*\right) - m^*\right]$ against m^*. Figure 4 illustrates this relationship and shows that $\left[E\left(m_t \middle| m_t \geq m^*\right) - m^*\right]$ is decreasing in m^*. Note that Figure 4 actually plots $\frac{E(m_t|m_t \geq m^*) - m^*}{\sigma_{m_t}}$ against $\frac{m^*}{\sigma_{m_t}}$ but since

$$\begin{aligned} E\left(m_t \middle| m_t \geq m^*\right) - m^* &= \sigma_{m_t} \frac{\phi\left(\frac{m^*}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*}{\sigma_{m_t}}\right)} - m^* \\ &= \sigma_{m_t}\left(\frac{\phi\left(\frac{m^*}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*}{\sigma_{m_t}}\right)} - \frac{m^*}{\sigma_{m_t}}\right) \\ &= \sigma_{m_t}\left(\frac{E\left(m_t \middle| m_t \geq m^*\right)}{\sigma_{m_t}} - \frac{m^*}{\sigma_{m_t}}\right) \end{aligned} \tag{5.17}$$

and σ_{m_t} is not influenced by m^*, then the figure still represents the relationship between $\left[E\left(m_t \middle| m_t \geq m^*\right) - m^*\right]$ and m^* when both axes are rescaled.

Remark 5.19 $\left[E\left(m_t \middle| m_t \geq m^*\right) - m^*\right]$ *is decreasing in* m^* *or* $\frac{\partial[E(m_t|m_t \geq m^*) - m^*]}{\partial m^*} < 0$ *for all* $m^* \in \mathbb{R}$.

To analyse the effects of h_1 and h_ε on $E\left(m_t \middle| m_t \geq m^*\right) - m^*$, equation (5.17) suggests that h_1 and h_ε influence $E\left(m_t \middle| m_t \geq m^*\right) - m^*$ via σ_{m_t} and also $\left(\frac{\phi\left(\frac{m^*}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*}{\sigma_{m_t}}\right)} - \frac{m^*}{\sigma_{m_t}}\right)$. Lemma 5.20 shows how h_1 and h_ε influence σ_{m_t}.

Lemma 5.20 *The standard deviation of* m_t, σ_{m_t}, *conditional on information at the time of product launching* ***decreases*** *if*

1. h_1 *increases*

2. h_ε *increases for* $t > 3 + \frac{h_1}{h_\varepsilon}$ *or* h_ε *decreases for* $t < 3 + \frac{h_1}{h_\varepsilon}$.

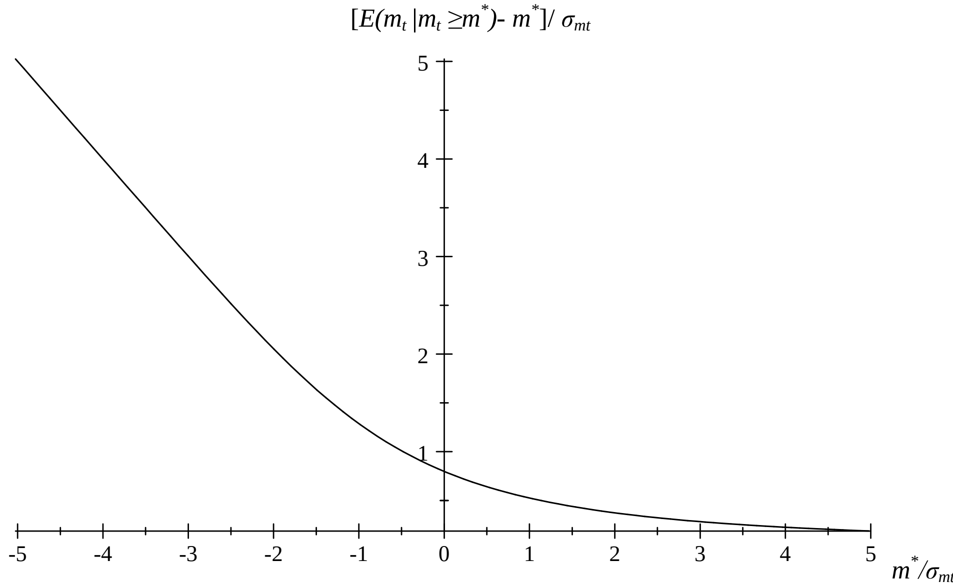

Figure 4: Plot of $\left[E\left(m_t\right| m_t \geq m^*\right) - m^*]$

Proof. For the first part, the partial derivative of σ_{m_t} with respect to h_1 is

$$\begin{aligned}
\frac{\partial \sigma_{m_t}}{\partial h_1} &= \frac{\partial}{\partial h_1}\left[(t-1)\left(\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right)^2 \frac{h_1 + h_\varepsilon}{h_1 h_\varepsilon}\right] \\
&= -\frac{1}{h_1^2}\frac{(t-1)\, h_\varepsilon}{\left((t-1)\, h_\varepsilon + h_1\right)^3}\left((t-1)\, h_\varepsilon^2 + 3h_\varepsilon h_1 + 2h_1^2\right) \\
&< 0
\end{aligned}$$

For the second part, the partial derivative of σ_{m_t} with respect to h_ε is given by

$$\frac{\partial \sigma_{m_t}}{\partial h_\varepsilon} = (t-1)\frac{(3-t)\, h_\varepsilon + h_1}{\left((t-1)\, h_\varepsilon + h_1\right)^3}$$

This expression is less than zero if and only if $(3-t)\, h_\varepsilon + h_1 < 0$ or $t > 3 + \frac{h_1}{h_\varepsilon}$. ■

The reason for Lemma 5.20.1 is that, when the precision of the initial prior belief is high, the posterior belief m_t is close to prior and then the variation or standard deviation of m_t is small. Lemma 5.20.2 suggests that as h_ε increases it causes σ_{m_t} to increase in the first few periods depending on the values of h_1 and h_ε but it reduces σ_{m_t} for large t. If the signal is very informative (high h_ε) and after $t = 3 + \frac{h_1}{h_\varepsilon}$ periods, the market will be able to predict the quality disturbance term γ more precisely and m_t is more closed to γ and is less variable (low σ_{m_t}).

Since σ_{m_t} is influenced by h_1 and h_ε and according to the plot from Figure 4 that $\left(\frac{\phi\left(\frac{m^*}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*}{\sigma_{m_t}}\right)} - \frac{m^*}{\sigma_{m_t}} \right)$ is decreasing in $\frac{m^*}{\sigma_{m_t}}$ (or increasing in σ_{m_t}), the relationship between $\left[E\left(m_t | m_t \geq m^*\right) - m^*\right] = \sigma_{m_t} \left(\frac{\phi\left(\frac{m^*}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*}{\sigma_{m_t}}\right)} - \frac{m^*}{\sigma_{m_t}} \right)$ and h_1, h_ε can be obtained. Remark 5.21 summarises this.

Remark 5.21 $E\left(m_t | m_t \geq m^*\right) - m^*$ ***decreases*** *if*

1. *h_1 increases and*

2. *h_ε increases for $t > 3 + \frac{h_1}{h_\varepsilon}$ or h_ε decreases for $t < 3 + \frac{h_1}{h_\varepsilon}$*

When h_1 increases, for the reason explained above, the standard deviation of m_t for the future t decreases. This means that the expectation of m_t given that $m_t \geq m^*$ or $E\left(m_t | m_t \geq m^*\right)$ must increase. Figure 5 shows the probability distribution functions of m_t with a different standard deviation ($\sigma_{m_t} = 1$ and 2). The vertical axis is drawn at the point where $m_t = m^*$ (arbitrary set $m^* = 1$ in the figure). The figure shows that areas to the right of the vertical axis have different expected values ($E\left(m_t | m_t \geq m^*\right) = 1.5251$ and 2.2822 respectively). When the precision of initial prior belief is higher, the future realisation of m_t tends to lie closer to m_1 and the firm cannot expect the high variation in m_t. Then, the expected return in period t, given that the product survives the discontinuation policy, will be lower.

Product discontinuation policy can be viewed as the policy for quality control. It is expressed in the form of minimal observed quality or reputation standard. Since the actual quality is unknown at the time of launching, the firm controls its product quality by monitoring consumers' experience from the product. This is reflected in the reputation of the product or the prior belief about the quality noise, m_t, in each period. A high m^* implies that the firm applies a rigid supervision on its product quality and it is more likely for the product to be discontinued. This idea is stated more formally in the next definition.

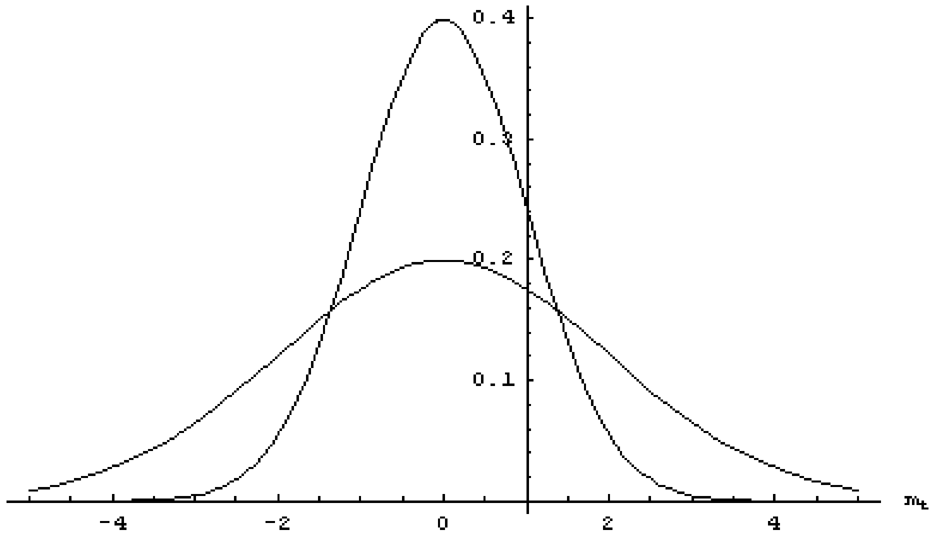

Figure 5: pdf functions of m_t with different σ_{m_t}

Definition 5.22 *Given any product discontinuation policy* $T = \min\left\{t > 1 : m_t \leq m^T\right\}$*, a product discontinuation policy* $T' = \min\left\{t > 1 : m_t \leq m^{T'}\right\}$ *is said to be* ***a more stringent (lenient) quality control*** *if* T' *calls to discontinue the product sooner (later) almost surely or* $m^{T'} > m^T$ $(m^{T'} < m^T)$.

Comparative static results can be obtained using the Implicit Function Theorem. From (5.16), define

$$F\left(m^*, \delta, h_1, h_\varepsilon\right) = \sum_{t=1}^{\infty} \kappa_t \delta^{t-1} \left(\sigma_{m_t} \frac{\phi\left(\frac{m^*}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*}{\sigma_{m_t}}\right)} - m^* \right)$$

Proposition 5.23 shows that, when the firm becomes more patient, the product has a shorter life almost surely.

Proposition 5.23 *As the discount factor* (δ) *increases, the optimal product discontinuation policy becomes more stringent.*

Proof. The partial derivative of $F\left(m^*, \delta, h_1, h_\varepsilon\right)$ with respect to the optimal discontinuation policy m^* is

$$F_{m^*} = \sum_{t=1}^{\infty} \delta^{t-1} \left[\kappa_t \frac{\partial \left[E\left(m_t \middle| m_t \geq m^* \right) - m^* \right]}{\partial m^*} + \left(E\left(m_t \middle| m_t \geq m^* \right) - m^* \right) \frac{\partial \kappa_t}{\partial \widetilde{m}_t} \frac{\partial \widetilde{m}_t}{\partial m^*} \right]$$
$$< 0$$

since $\left(E\left(m_t \middle| m_t \geq m^* \right) - m^* \right) > 0$ for all $m^* \in \mathbb{R}$ and from Remark 5.18.1 and Remark 5.19, $\frac{\partial \kappa_t}{\partial \widetilde{m}_t} \frac{\partial \widetilde{m}_t}{\partial m^*} < 0$ and $\frac{\partial [E(m_t|m_t \geq m^*) - m^*]}{\partial m^*} < 0$. The partial derivative of $F\left(m^*, \delta, h_1, h_\varepsilon\right)$ with respect to δ is

$$F_\delta = \sum_{t=1}^{\infty} (t-1) \, \delta^{t-2} \kappa_t \left[E\left(m_t \middle| m_t \geq m^* \right) - m^* \right]$$
$$> 0$$

By the Implicit Function Theorem, $\frac{dm^*}{d\delta} = -\frac{F_\delta}{F_{m^*}} > 0$. ■

The result shown in Proposition 5.23 seems counter-intuitive but this reveals an important implication: a patient firm will be impatient with its low quality products. When the firm is patient, it puts greater weight on futures, supplies better quality and gets a higher expected revenue and hence a higher expected per period revenue. The firm will stop the product in the period when the period revenue falls below the expected per period revenue, since the firm knows that it can get a higher revenue on average by launching the new product. Therefore, the product will be discontinued earlier almost surely.

Proposition 5.24 *As the precision of initial prior belief (h_1) increases, the optimal product discontinuation policy becomes more lenient.*

Proof. The partial derivative of $F\left(m^*, \delta, h_1, h_\varepsilon\right)$ with respect to the precision of initial prior belief h_1 is

$$F_{h_1} = \sum_{t=1}^{\infty} \delta^{t-1} \left[\kappa_t \frac{\partial \left[E\left(m_t \middle| m_t \geq m^* \right) - m^* \right]}{\partial h_1} + \left(E\left(m_t \middle| m_t \geq m^* \right) - m^* \right) \frac{\partial \kappa_t}{\partial \widetilde{m}_t} \frac{\partial \widetilde{m}_t}{\partial h_1} \right]$$

From Lemma 5.20, $\frac{\partial [E(m_t|m_t \geq m^*) - m^*]}{\partial h_1} < 0$ and Remark 5.18, $\frac{\partial \kappa_t}{\partial \widetilde{m}_t} \frac{\partial \widetilde{m}_t}{\partial h_1} < 0$, it follows that $F_{h_1} < 0$. It was shown in the proof of Proposition 5.23 that $F_{m^*} < 0$. Therefore, by the Implicit Function Theorem, $\frac{dm^*}{dh_1} = -\frac{F_{h_1}}{F_{m^*}} < 0$ ■

As the initial prior belief about the product quality noise becomes more precise, the expected return from the product decreases for two reasons. First, the probability that the product will survive the product discontinuation policy is lower because of the tendency that m_t will lie closer to m_1. Second, the higher tendency that m_t will lie closer to m_1 reduces the expected return from the product given that it survives the discontinuation policy. The smaller expected return from the product lowers the condition for which the product is discontinued.

The effect of h_ε on m^* is ambiguous. A higher h_ε clearly raises the probability that the product will be continued in every period. However, its effect on the expected profit, given that the product survives, is unclear as $E\left(m_t|\, m_t \geq m^*\right) - m^*$ increases for the first few periods but declines afterwards. The net effect relies on the parameter values of the model.

6 Quality Choices

The optimal product discontinuation policy and the value function derived in Corollary 5.17 depend upon the optimal level of quality choice, q^*. This section derives a condition for which the optimal quality is determined. Note that the optimal quality also depends on the optimal discontinuation policy and that both must be solved simultaneously using the condition shown in Corollary 5.17 and the one derived in this section.

A firm chooses a quality choice that maximises the product's lifetime expected profit on the basis that an optimal product discontinuation policy will be implemented after the product is launched. For any period t, the probability that a product with quality q will be continued under the implementation of an optimal discontinuation policy is $\kappa_t\left(q, \lambda^*\right)$. The expected profit from a product with quality q is given by

$$\sum_{t=1}^{\infty} \delta^{t-1} \kappa_t\left(q, \lambda^*\right)\left[E\left(p_t\left|m_t \geq m^*\left(\lambda^*, q^*\right)\right.\right) - C\left(q\right)\right]$$

where p_t is consumers' willingness to pay defined in (3.7). The expectation is now conditional on events where the product survives the optimal discontinuation policy. Using (3.7), the above expression can be rewritten as

$$\sum_{t=1}^{\infty} \delta^{t-1} \kappa_t\left(q, \lambda^*\right)\left[q^* + E\left(m_t\left|m_t \geq m^*\left(\lambda^*, q^*\right)\right.\right) - C\left(q\right)\right]$$

The firm's quality choice problem is

$$\sup_q \sum_{t=1}^{\infty} \delta^{t-1} \kappa_t \left(q, \lambda^*\right) \left[q^* + E\left(m_t \left| m_t \geq m^*\left(\lambda^*, q^*\right)\right.\right) - C\left(q\right)\right]$$

From the first part of Lemma 5.15, the problem can be expressed as follows,

$$\sup_q \sum_{t=1}^{\infty} \delta^{t-1} \kappa_t \left(q, \lambda^*\right) \left[q^* + \mu_t\left(q\right) + \sigma_{m_t} \frac{\phi\left(\frac{m^*(\lambda^*, q^*) - \mu_t(q)}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*(\lambda^*, q^*) - \mu_t(q)}{\sigma_{m_t}}\right)} - C\left(q\right)\right] \tag{6.1}$$

where $\mu_t\left(q\right) = \frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1}\left(q - q^*\right)$, $\sigma_{m_t}^2 = (t-1)\left[\frac{h_\varepsilon}{(t-1)h_\varepsilon + h_1}\right]^2 \frac{h_1 + h_\varepsilon}{h_1 h_\varepsilon}$, $\phi\left(\cdot\right)$ and $\Phi\left(\cdot\right)$ are the standard normal pdf. and cdf. respectively.

The FOC for the quality problem (6.1) is presented in the following proposition

Proposition 6.1 *For any $\lambda^* \in \mathbb{R}$ and any optimal product discontinuation policy $T^* = \min\left\{t > 1 : m_t \leq m^*\left(\lambda^*, q^*\right)\right\}$, the optimal product quality, q^*, satisfies the following FOC*

$$\begin{aligned}\sum_{t=1}^{\infty} \delta^{t-1} \kappa_t\left(\lambda^*\right) \left(\mu_t' - \mu_t' \frac{\phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)} \left(\frac{\phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)} - \frac{m^*\left(\lambda^*, q^*\right)}{\sigma_{m_t}}\right) - C'\left(q^*\right)\right) \\ + \sum_{t=1}^{\infty} \delta^{t-1} \left.\frac{\partial}{\partial q} \kappa_t\left(q, \lambda^*\right)\right|_{q=q^*} \left(q^* + \sigma_{m_t} \frac{\phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)} - C\left(q^*\right)\right) = 0 \end{aligned} \tag{6.2}$$

where $\kappa_t\left(\lambda^*\right) = \kappa_t\left(q, \lambda^*\right)$ is evaluated at $q = q^*$ and $\mu_t' = \frac{\partial}{\partial q}\mu_t\left(q\right) = \frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon + h_1}$.

Proof. According to product rule, the FOC for the problem (6.1) is

$$\begin{aligned}\sum_{t=1}^{\infty} \delta^{t-1} \kappa_t\left(\lambda^*\right) \left(\mu_t' - \mu_t' \left(\frac{\phi'\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)\left(1 - \Phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)\right) + \phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)\phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)}{\left(1 - \Phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)\right)^2}\right)\right. \\ \left. - C'\left(q^*\right)\right) + \sum_{t=1}^{\infty} \delta^{t-1} \left.\frac{\partial}{\partial q} \kappa_t\left(q, \lambda^*\right)\right|_{q=q^*} \left(q^* + \sigma_{m_t} \frac{\phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)}{1 - \Phi\left(\frac{m^*(\lambda^*, q^*)}{\sigma_{m_t}}\right)} - C\left(q^*\right)\right) = 0\end{aligned}$$

using the fact that for the standard normal distribution, $\phi'\left(v\right) = -v\phi\left(v\right)$, and re-scaled yield

$$\sum_{t=1}^{\infty}\delta^{t-1}\kappa_t\left(\lambda^*\right)\left(\mu_t'-\mu_t'\frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}\left(\frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}-\frac{m^*\left(\lambda^*,q^*\right)}{\sigma_{m_t}}\right)-C'\left(q^*\right)\right)$$
$$+\sum_{t=1}^{\infty}\delta^{t-1}\left.\frac{\partial}{\partial q}\kappa_t\left(q,\lambda^*\right)\right|_{q=q^*}\left(q^*+\sigma_{m_t}\frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}-C\left(q^*\right)\right)=0$$

■

The Proposition 6.1 suggests that there are two channels by which quality choice can affect the expected return. The first term of the FOC (6.2) is the standard career concerns mechanism. The firm has an incentive to increase its product quality in order to mislead consumers that the product has a high unobservable quality because this will increase consumers' willingness to pay and the firm's profits. The second term of the FOC (6.2) represents the second incentive for the firm to increase quality: higher quality biases consumer inference about unobservable quality noise, and increases the probability that the product will be continued, which in turn increases expected revenue. This additional incentive to increase quality is the extension of the prototype career concerns model and occurs in the context where the relationship between principal and agent can be terminated by the realisation of the random variable. In both mechanisms the firm or agent exerts a greater effort to mislead consumer or principal inference and this bias has two effects. Firstly, it encourages principal to believe that the agent is of the better type and this will results in higher returns to the agent. Secondly, when the probability of relationship termination depends on belief about the agent's type or the realisation of the random variable which the agent can influence, the agent will exert greater effort to reduce this probability. The higher the chance that the relationship will be continued means the higher the expected return from the relationship.

The FOC (6.2) in Proposition 6.1 is a generalisation of the FOC (4.2) derived in Section 4.2 where product discontinuation policy is not considered. One can obtain condition (4.2) by setting m^* in (6.2) approaches $-\infty$ (where the product will never be discontinued). To verify this, rewrite (6.2) as the following,

$$\sum_{t=1}^{\infty}\delta^{t-1}\kappa_t\left(\lambda^*\right)\left[\mu_t'-\mu_t'\left(\left(\frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}\right)^2-\frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)m^*\left(\lambda^*,q^*\right)}{\sigma_{m_t}\left(1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)\right)}\right)-C'\left(q^*\right)\right]$$
$$+\sum_{t=1}^{\infty}\delta^{t-1}\left.\frac{\partial}{\partial q}\kappa_t\left(q,\lambda^*\right)\right|_{q=q^*}\left(q^*+\sigma_{m_t}\frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}-C\left(q^*\right)\right)=0$$

since for the standard normal distribution, $\phi'(v)=-v\phi(v)$, the expression becomes

$$\sum_{t=1}^{\infty}\delta^{t-1}\kappa_t\left(\lambda^*\right)\left[\mu_t'-\mu_t'\left(\left(\frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}\right)^2+\frac{\phi'\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{\sigma_{m_t}\left(1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)\right)}\right)-C'\left(q^*\right)\right]$$
$$+\sum_{t=1}^{\infty}\delta^{t-1}\left.\frac{\partial}{\partial q}\kappa_t\left(q,\lambda^*\right)\right|_{q=q^*}\left(q^*+\sigma_{m_t}\frac{\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}{1-\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)}-C\left(q^*\right)\right)=0 \quad (6.3)$$

as $m^*\to-\infty$, $\kappa_t(\lambda^*)\to 1$ or the product will never be discontinued and $\left.\frac{\partial}{\partial q}\kappa_t(q,\lambda^*)\right|_{q=q^*}\to 0$. Since $\phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)$ and $\Phi\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)$ approach 0 as $m^*\to-\infty$, the second term in LHS of (6.3) vanishes. and because the slope of standard normal pdf, $\phi'\left(\frac{m^*(\lambda^*,q^*)}{\sigma_{m_t}}\right)$, converges to 0 as $m^*\to-\infty$. Therefore, as $m^*\to-\infty$, equation (6.3) collapses to

$$\sum_{t=1}^{\infty}\delta^{t-1}\left[\mu_t'-C'\left(q^*\right)\right]=0$$

but since $\mu_t'=\frac{\partial}{\partial q}\mu_t(q)=\frac{(t-1)h_\varepsilon}{(t-1)h_\varepsilon+h_1}$, this is identical to (4.2).

Note that, when the product is never discontinued ($m^*\to-\infty$), the incentive to raise quality in order to increase the probability of product continuation disappears. Only the incentive to bias consumers' learning that increases the consumers' valuation and the firm's expected return remains; this is the standard career concerns mechanism.

7 Conclusion

This paper has considered and investigated quality choice and product discontinuation problems under a moral hazard problem where information about product quality is incomplete but symmetric; information about true product quality is unknown to both consumers and firm. The firm's intended quality choice is difficult to control and subject to random shocks. This shock can be interpreted as a match between consumer's preference and product characteristics. The true quality is unobservable, and consumers imperfectly report their (noisy) experiences from the product to the next generation of consumers. Reports serve as noisy signals of the true quality. The market forms beliefs about a product's true quality based on the observed sequence of noisy signals or a product's reputation.

The paper started by introducing a career concerns type of model and investigating relationships between optimal quality choice and various variables. It was found that a firm offers a better quality product when the product remains longer on the market (i.e. has a longer life), the firm cares about long term relationship (i.e. has a higher discount factor), the precision of initial prior belief is low, and the signal is very informative. The mechanism behind these results is the standard career concerns' incentive (Holmström 1982). According to this model, the firm has an incentive to signal jam the consumer's learning process since consumers cannot tell whether perceived quality is due to characteristics that match their preference (i.e. product is of better type) or just because of the imperfect communication of experiences (i.e. high realisations of experience or report shocks). High quality is intended to bias consumer belief that a product fulfils their preferences.

The paper then moved on to consider the case in which a firm can decide to discontinue a product before its true quality is completely revealed and then relaunch a new product activating a new market learning process. The scenario in which the product discontinuation period is predetermined or fixed in advance was considered in the first instance. Here it was shown that longer product life increases the expected return, and that it is better for a firm to have one product with an infinite life than many products with a shorter life. In the deterministic stopping period case, a firm therefore never opts for a predetermined discontinuation period.

The deterministic stopping period assumption was then relaxed. The case of a firm choosing to discontinue and relaunch a product whenever and as often as it wants was then considered. In this context, the firm can choose to discontinue the product after its reputation for that period is realised. The paper proved that optimal product discontinuation policies or strategies exist and the optimal discontinuation policy was then derived. Comparative static analysis has been carried out and demonstrated that when a firm cares about the long-term relationship, it applies a more stringent quality-control discontinuation policy and the product tends to be discontinued sooner almost surely. This is also true when the precision of initial prior belief is low. The informativeness of the signal has an ambiguous effect on the policy.

The optimal condition for quality with discontinuation policy was derived and has been shown to be a generalisation of the quality choice first order condition when discontinuation decision is not allowed. Under the presence of discontinuation policy, a quality choice chosen by the firm, again, misleads consumers that the product possesses favourable characteristics. However, as opposed to standard career concerns models without product discontinuation, this affects the expected return by two means. The first is through increased consumer willingness to pay for the product; the standard career concerns mechanism. And secondly, the higher quality will increase chances of surviving the discontinuation policy, enhance the probability of product continuation and raise firm's expected return.

When the firm relaunches the product, the firm's "type" is re-drawn independently from the distribution identical to all of the previous products. In this case, consumers cannot use information about the previous products evaluate the new product's true quality. However, in the real world consumers have diverse expectations about the quality of new products launched by different firms or brands believing them to possess different characteristics or types that are permanent or at least stable. It would be interesting for future research to incorporate into this model the adverse selection aspect when firms use quality to signal their types. Another interesting dimension would be the role of advertising that, in addition to quality choice, could confuse and mislead consumer inference. This would offer an insight into the firm's dilemma between enhancing quality and intensifying advertising.

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